ZERO-DIVISOR AND IDEAL-DIVISOR GRAPHS OF COMMUTATIVE RINGS

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Abstract. For a commutative ring R, we can form the zero-divisor graph $\Gamma(R)$ or the ideal-divisor graph $\Gamma_I(R)$ with respect to an ideal I of R. We consider the diameters of direct products of zero-divisor and ideal-divisor graphs.

1. Introduction and Definitions

The concept of a zero-divisor graph was first introduced in 1988 in [3] by Beck, who was interested in the coloring of the zero-divisor graph. However, we use the slightly altered definition of a zero-divisor graph offered by Anderson and Livingston in [1]. Anderson and Livingston also proved that the diameter of $\Gamma(R)$ is less than or equal to three for all commutative rings R. This knowledge of small diameter of zero-divisor graphs led Axtell, Stickles, and Warfel to find necessary and sufficient conditions for the direct product of two commutative rings R_1 and R_2 to have various diameters in [2].

Given a commutative ring R, recall that the set of zero-divisors $Z(R)$ is the set $\{x \in R \mid \text{there exists } y \in R^* \text{ such that } xy = 0\}$, where $R^* = R - \{0\}$. Also, $Z(R)^* = Z(R) - \{0\}$. Finally, we define regular elements to be $\text{reg}(R) = R - Z(R)$ and the annihilator of a zero-divisor x to be $\text{ann}(x) = \{y \in Z(R)^* \mid xy = 0\}$. We can define the **zero-divisor graph** of R, $\Gamma(R)$, as follows: $x \in \Gamma(R)$ if and only if $x \in Z(R)^*$, and $x, y \in \Gamma(R)$ are adjacent if and only if $xy = 0$. Furthermore, we can define the diameter of any graph $\text{diam}(\Gamma) = \max\{d(x, y) \mid x \text{ and } y \text{ are distinct vertices of } \Gamma(R)\}$.

For the sake of generalization we can think of $\{0\}$ as an ideal of the ring R. Given a commutative ring R and *any* ideal I of R, we define the **ideal-divisors** of R with respect to I as $Z_I(R) = \{x \in R \mid \text{there exists } y \in R - I \text{ such that } xy \in I\}$. Furthermore, we can define $Z_I(R)^*$ as $Z_I(R) - I$. Also, $\text{reg}_I(R) = R - Z_I(R)$ and $\text{ann}_I(x) = \{y \in Z_I(R)^* \mid xy \in I\}$. We can define the **ideal-divisor graph** of R, $\Gamma_I(R)$, by letting x be an element of $\Gamma_I(R)$ if and only if x is an element of $Z_I(R)^*$; x and y in $\Gamma_I(R)$ are adjacent if and only if xy is an element of I.

The ideal-divisor graph was first discussed by Redmond in [4]. He was able to generalize many of the concepts and theorems of the zero-divisor graph to the ideal-divisor graph. In particular, Redmond showed that the diameter of $\Gamma_I(R)$ is less than or equal to three for all commutative rings R and I an arbitrary ideal of R.

In this paper we complete the classification done by Axtell, Stickles, and Warfel in [2] to include direct products of commutative rings that have diameter zero and generalize the entire classification to ideal-divisor graphs.

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2. Ideal-Divisor Graphs of Direct Products

We begin with some necessary lemmas generalized from similar lemmas used by Axtell, Stickles, and Warfel.

Lemma 2.1. *Let R_1 and R_2 be rings with ideals I_1 and I_2, respectively, with $Z_{I_1}(R_1)^* \neq \varnothing$ or $Z_{I_2}(R_2)^* \neq \varnothing$ (or both). If $\text{reg}_{I_1}(R_1) \neq \varnothing$ and $\text{reg}_{I_2}(R_2) \neq \varnothing$, then $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 3$.*

Proof. Without loss of generality, let $a \in Z_{I_1}(R_1)^*$. Then there exists $b \in Z_{I_1}(R_1)^*$ such that $ab \in I_1$. Let $r_1 \in \text{reg}_{I_1}(R_1)$, $r_2 \in \text{reg}_{I_2}(R_2)$. Then

$$(r_1, 0) - (0, r_2) - (a, 0) - (b, r_2)$$

is a path of length 3. Assume there is a shorter path from $(r_1, 0)$ to (b, r_2). Clearly, $(r_1, 0)(b, r_2) \notin I_1 \times I_2$, so there is no path of length 1. Assume there exists $(x, y) \in Z_{I_1 \times I_2}(R_1 \times R_2)^*$ such that $(x, y)(r_1, 0), (x, y)(b, r_2) \in I_1 \times I_2$. Then $xr_1 \in I_1$ and $yr_2 \in I_2$, which gives $x \in I_1$ and $y \in I_2$, a contradiction. Hence, $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 3$. □

Lemma 2.2. *If $R_1 = I_1$ and $\text{diam}(\Gamma_{I_2}(R_2)) > 0$, then $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = \text{diam}(\Gamma_{I_2}(R_2))$.*

Proof. Let $R_1 = I_1$ and let $\text{diam}(\Gamma_{I_2}(R_2)) > 0$.

- Suppose $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = n > \text{diam}(\Gamma_{I_2}(R_2))$ such that $n = 2$ or 3. Then $\exists (a_0, x_0), (a_1, x_1), ..., (a_n, x_n) \in Z_{I_1 \times I_2}(R_1 \times R_2)^*$ such that $(a_0, x_0) - (a_1, x_1) - ... - (a_n, x_n)$ is a minimal path. Then $x_0 - x_1 - ... - x_n$, but since $n > \text{diam}(\Gamma_{I_2}(R_2))$, $x_0 - x_1 - ... - x_n$ must not be a minimal path. This can happen in two ways:
  - If $\exists i, j$ such that $0 \leq i < j \leq n$, $j \neq i + 1$, and $x_i - x_j$. Then $(a_i, x_i) - (a_j, x_j)$, a contradiction of $(a_0, x_0) - (a_1, x_1) - ... - (a_n, x_n)$being a minimal path.
  - If $n = 3$ and $\exists y \in Z_{I_2}(R_2)^*$ such that $y \notin \{x_i \mid 0 \leq i \leq n\}$ and $x_0 - y - x_n$. Then $(a_0, x_0) - (0, y) - (a_n, x_n)$, a contradiction of $(a_0, x_0) - (a_1, x_1) - ... - (a_n, x_n)$being a minimal path.

  So $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) \leq \text{diam}(\Gamma_{I_2}(R_2))$.
- Suppose $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) < \text{diam}(\Gamma_{I_2}(R_2)) = n$ such that $1 \leq n \leq 3$. Then $\exists x_0, x_1, ..., x_n \in Z_{I_2}(R_2)^*$ such that $x_0 - x_1 - ... - x_n$ is a minimal path. Since $R_1 = I_1$, $\forall a_0, a_1, ..., a_n \in R_1$, $(a_0, x_0) - (a_1, x_1) - ... - (a_n, x_n)$ is a minimal path of length n, a contradiction. So $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) \geq \text{diam}(\Gamma_{I_2}(R_2))$.

Thus $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = \text{diam}(\Gamma_{I_2}(R_2))$. □

Lemma 2.3. *(Analogue to ASW Lemma 2.2) Let R be a commutative ring with ideal I. If $R = Z_I(R)$ and $\text{diam}(\Gamma_I(R)) = 1$, then $R^2 \subseteq I$.*

Proof. Note that for all $x, y \in R$ with $x \neq y, xy \in I$ because $\text{diam}(\Gamma_I(R)) = 1$. Assume that $a^2 \notin I$ for some $a \in R$. Let $b \in Z_I(R)^*$ with $b \neq a$. Observe that $a + b \neq a$. Since $R = Z_I(R)$ and $\text{diam}(\Gamma_I(R)) = 1$, we have $ab \in I$. So, $a(a + b) = a^2 + ab \in I$, which implies $a^2 \in I$, a contradiction. □

Theorem 2.4. *Let R_1, R_2 be commutative rings such that $\text{diam}(\Gamma(R_1)) = \text{diam}(\Gamma(R_2)) = 0$. Then*

(i) $\text{diam}(\Gamma(R_1 \times R_2)) = 0$ *if and only if (without loss of generality)* $R_1 = \{0\}$.
(ii) $\text{diam}(\Gamma(R_1 \times R_2)) = 1$ *if and only if* $|R_1| = |R_2| = 2$ *and* $R_1 \cong R_2$.
(iii) $\text{diam}(\Gamma(R_1 \times R_2)) = 3$ *if and only if* $R_1 \neq Z(R_1), R_2 \neq Z(R_2)$*, and (without loss of generality)* $|Z(R_1)^*| = 1$.
(iv) $\text{diam}(\Gamma(R_1 \times R_2)) = 2$ *otherwise.*

Proof. (i) ($\Rightarrow$) Assume for contradiction that $\text{diam}(\Gamma(R_1 \times R_2)) = 0$ and $R_1, R_2 \neq \{0\}$. Then $\exists x \in R_1^*, y \in R_2^*$. Now $(x, 0)$ and $(0, y)$ are two adjacent vertices in $\Gamma(R_1 \times R_2)$, so $\text{diam}(\Gamma(R_1 \times R_2)) > 0$.
($\Leftarrow$) Assume without loss of generality that $R_1 = \{0\}$. Then $R_1 \times R_2 \cong R_2$, so $\text{diam}(\Gamma(R_1 \times R_2)) = 0$.

(ii) ($\Rightarrow$) Assume $\text{diam}(\Gamma(R_1 \times R_2)) = 1$.
- Assume without loss of generality $|R_1| \neq 2$. If $|R_1| = 1$, then $R_1 = \{0\}$, so $\text{diam}(\Gamma(R_1 \times R_2)) = 0$ by part (i). Thus assume $|R_1| \geq 3$. Thus we have distinct $x, y \in R_1^*$. Note that $xy \neq 0$ because $\text{diam}(\Gamma(R_1)) = 0$. Also, note that $|R_2| > 1$ by part (i) also. Let $z \in R_2^*$. Now, we have path
$$(x, 0) - (0, z) - (y, 0)$$
which is minimal since $xy \neq 0$. Thus $\text{diam}(\Gamma(R_1 \times R_2)) \geq 2$.
- Assume $R_1 \ncong R_2$. By argument above, $|R_1| = |R_2| = 2$. Recall that there are only two rings of order two: $\mathbb{Z}_2$ and $2\mathbb{Z}_4$. Since $\text{diam}(\Gamma(\mathbb{Z}_2 \times 2\mathbb{Z}_4)) = 2$, we have a contradiction. Thus $R_1 \cong R_2$.

($\Leftarrow$) Assume $|R_1| = |R_2| = 2$ and $R_1 \cong R_2$. Since there are only two rings of order two, and $\text{diam}(\Gamma(2\mathbb{Z}_4 \times 2\mathbb{Z}_4)) = 1 = \text{diam}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = 1$, then $\text{diam}(\Gamma(R_1 \times R_2)) = 1$.

(iii) ($\Rightarrow$) Assume $\text{diam}(\Gamma(R_1 \times R_2)) = 3$.
- Assume without loss of generality that $Z(R_1) = R_1$. Also $|R_1| > 1$ by part (i). Thus, since $\text{diam}(\Gamma(R_1)) = 0$, $R_1 \cong 2\mathbb{Z}_4$. Then every vertex in $\Gamma(R_1 \times R_2)$ is adjacent to $(x, 0)$, where $x \in R_1^*$. Then $\text{diam}(\Gamma(R_1 \times R_2)) \leq 2$. This is a contradiction, of $\text{diam}(\Gamma(R_1 \times R_2)) = 3$, so $R_1 \neq Z(R_1)$ and $R_2 \neq Z(R_2)$.
- Assume $\Gamma(R_1) = \Gamma(R_2) = K_0$, the graph with no vertices. Thus $Z(R_1)^* = Z(R_2)^* = \varnothing$. Thus every vertex in $\Gamma(R_1 \times R_2)$ must be of the form $(x, 0)$ or $(0, y)$, where $x \in \text{reg}(R_1), y \in \text{reg}(R_2)$. Thus $\Gamma(R_1 \times R_2)$ is complete bipartite, so $\text{diam}(\Gamma(R_1 \times R_2)) \leq 2$.

($\Leftarrow$) By Lemma 2.1.

(iv) Follows from (i), (ii), (iii). □

3. Ideal-Divisor Graphs of Direct Products (Diameter Zero By N)

Lemma 3.1. *Let* $\Gamma_I(R)$ *be the ideal-divisor graph of* R *with respect to an ideal* I *such that* $\Gamma_I(R)$ *has exactly one vertex. Then* $I = \{0\}$.

Proof. Assume $\Gamma_I(R)$ as above. Then $\exists x \in Z_I(R)^*$ such that $x^2 \in I$. Let $i \in I$. Now $(x + i)x = x^2 + ix \in I$. Thus, since there is only one vertex in $\Gamma_I(R)$, we know that $x + i = x$, so $i = 0$. Thus $I = \{0\}$. □

Lemma 3.2. *(Analogue to ASW Lemma 2.3) Let* R *be a commutative ring with ideal* I *such that* $\text{diam}(\Gamma_I(R)) = 2$*. Suppose* $Z_I(R)$ *is a (not necessarily proper) subring of* R*. Then for all* $x, y \in Z_I(R)$*, there exists a* $z \notin I$ *such that* $xz, yz \in I$.

Proof. Let $x, y \in Z_I(R)$. If $x \in I$, then simply choose z such that $yz \in I$. We can similarly find z if $y \in I$ or if $x = y$. Thus we assume x, y are distinct and not in I. Since $\operatorname{diam}(\Gamma_I(R)) = 2$, if $xy \notin I$, we know that there exists some $z \in Z_I(R)^*$ such that $xz, yz \in I$. So assume $xy \in I$. If $x^2 \in I$, then clearly we can choose $z = x$. A similar situation exists if $y^2 \in I$. Therefore we may also assume that $x^2, y^2 \notin I$.

Let $X = \{x' \in Z_I(R)^* \mid xx' \in I\}$ and $Y = \{y' \in Z_I(R)^* \mid yy' \in I\}$. Note that since $x \in Y$ and $y \in X$, these sets are not empty. Furthermore, if $X \cap Y \neq \varnothing$, then choosing $z \in X \cap Y$ will suffice.

Thus assume that $X \cap Y = \varnothing$. Consider the element $x + y$. Obviously $x + y \neq x$ and $x + y \neq y$. If $x + y \in I$, then $x(x + y) \in I$ which implies that $x^2 \in I$. Thus $x + y \notin I$. Thus $x + y \in Z_I(R)^*$ since $Z_I(R)$ is a subring. Furthermore, since $x^2, y^2 \notin I$, we know that $x + y \notin X$ and $x + y \notin Y$. But the diameter of $\Gamma_I(R)$ is 2, so we know that we can find some $w \in X$ such that $xw \in I$, $w(x+y) \in I$. But then $w(x+y) \in I$ which implies that $wy \in I$, so $w \in X \cap Y$, which is a contradiction. □

Theorem 3.3. *Let R_1, R_2 be commutative rings and I_1, I_2 be their respective ideals such that* $\operatorname{diam}(\Gamma_{I_1}(R_1)) = \operatorname{diam}(\Gamma_{I_2}(R_2)) = 0$. *Then*

(i) $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 0$ *if and only if (without loss of generality) $I_1 = R_1$ and either $R_1 = \{0\}$ or I_2 is a prime ideal*

(ii) $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 1$ *if and only if (without loss of generality) $I_1 = \{0\}$ and either $Z_{I_1}(R_1)^* \neq \varnothing$ with $I_2 = R_2 \neq \{0\}$ or $R_1 \cong R_2$ with $|R_1| = 2$*

(iii) $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 3$ *if and only if $R_1 \neq Z_{I_1}(R_1), R_2 \neq Z_{I_2}(R_2)$ and (without loss of generality) $|Z_{I_1}(R_1)^*| = 1$.*

(iv) $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 2$ *otherwise.*

Proof. (i) ($\Rightarrow$) Assume $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 0$.

Now, if $\operatorname{reg}_{I_1}(R_1), \operatorname{reg}_{I_2}(R_2) \neq \varnothing$, then $\exists x \in \operatorname{reg}_{I_1}(R_1), y \in \operatorname{reg}_{I_2}(R_2)$. Then $(x, 0)(0, y) = 0 \in I_1 \times I_2$, which is a contradiction of $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 0$. Thus, without loss of generality, $\operatorname{reg}_{I_1}(R_1) = \varnothing$, so $R_1 = I_1$.

- If $\Gamma_{I_1 \times I_2}(R_1 \times R_2) = K_0$, then $I_1 \times I_2$ is a prime ideal. Now, consider $ab \in I_2$. Then $(0, a)(0, b) \in I_1 \times I_2$. Since $I_1 \times I_2$ is prime, $(0, a) \in I_1 \times I_2$ or $(0, b) \in I_1 \times I_2$.Thus $a \in I_2$ or $b \in I_2$. Thus I_2 is a prime ideal.
- If $\Gamma_{I_1 \times I_2}(R_1 \times R_2)$ has one vertex, then $I_1 \times I_2 = \{0\}$, by Lemma 3.1. Then $I_1 = I_2 = \{0\}$, and $\Gamma_{I_1 \times I_2}(R_1 \times R_2) = \Gamma(R_1 \times R_2)$. By Theorem 2.4, then without loss of generality, $R_1 = \{0\}$.

($\Leftarrow$) Assume $I_1 = R_1$ and either $R_1 = \{0\}$ or I_2 is a prime ideal.

- Let $I_1 = R_1 = \{0\}$. If $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) > 0, \exists (0, y_1), (0, y_2) \notin I_1 \times I_2$ such that $(0, y_1)(0, y_2) \in I_1 \times I_2$. Then $y_1, y_2 \notin I_2$ and $y_1 y_2 \in I_2$, so $\operatorname{diam}(\Gamma_{I_2}(R_2)) > 0$. This is a contradiction, so $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 0$.
- Let $I_1 = R_1$, and I_2 be a prime ideal of R_2. If $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) > 0$, then there exist distinct $(x_1, y_1), (x_2, y_2) \notin I_1 \times I_2$ such that $(x_1, y_1)(x_2, y_2) \in I_1 \times I_2$. Since $R_1 = I_1$, we have $y_1, y_2 \notin I_2$ and $y_1 y_2 \in I_2$. Thus I_2 is not a prime ideal. This is a contradiction, so $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 0$.

(ii) ($\Rightarrow$) Assume $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 1$.

- Assume $\Gamma_{I_1}(R_1) = \Gamma_{I_2}(R_2) = K_0$. Since $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 1, \exists$ vertices $(x, 0), (0, y) \in Z_{I_1 \times I_2}(R_1 \times R_2)^*$. Thus $\exists x \in \operatorname{reg}_{I_1}(R_1), y \in \operatorname{reg}_{I_2}(R_2)$. For all $i \in I_1, (x + i, 0)(0, y) \in I_1 \times I_2$, and $(x + i, 0)(x, 0) \notin$

$I_1 \times I_2$. This implies that $(x+i, 0) = (x, 0)$, so $i = 0$ and thus $I_1 = \{0\}$. Similarly, $I_2 = \{0\}$. By Theorem 2.4, $|R_1| = |R_2| = 2$ and $R_1 \cong R_2$.

- Assume $\Gamma_{I_1}(R_1) = K_1$ and $\Gamma_{I_2}(R_2) = K_0$. By Lemma 3.1, $I_1 = \{0\}$. Let $x \in Z_{I_1}(R_1)^*$ (x must exist since $\Gamma_{I_1}(R_1) = K_1$). If $R_2 \neq I_2, \exists y \in \text{reg}_{I_2}(R_2)$. Then $(x, 0)(x, y) \in I_1 \times I_2$ and $(x, 0)(0, y) \in I_1 \times I_2$, but $(x, y)(0, y) \notin I_1 \times I_2$. Thus $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) \geq 2$. So $I_2 = R_2$. Also note that $I_2 \neq \{0\}$ (since $I_2 = \{0\}$ would imply $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 0$ by part (i) above).
- Assume $\Gamma_{I_1}(R_1) = \Gamma_{I_2}(R_2) = K_1$. By Lemma 3.1, $I_1 = I_2 = \{0\}$, so $|R_1| = |R_2| = 2$ and $R_1 \cong R_2$ by Theorem 2.4.

($\Leftarrow$)

- Assume that $I_1 = \{0\}$ and $R_1 \cong R_2$ with $|R_1| = 2$. Thus $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 1$ by Theorem 2.4.
- Assume that $I_1 = \{0\}$ and $Z_{I_1}(R_1)^* \neq \varnothing$ with $I_2 = R_2 \neq \{0\}$. Let $x \in Z_{I_1}(R_1)^*$. Thus, for all vertices $(x, y_1), (x, y_2) \in Z_{I_2 \times I_2}(R_1 \times R_2)^*$ (we know $y_1, y_2 \in I_2$, so the first entry must indeed be x), it's clear that $(x, y_1)(x, y_2) \in I_1 \times I_2$. Also, there are at least two vertices since $I_2 \neq \{0\}$, so $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 1$.

(iii) ($\Rightarrow$)Assume $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 3$.

- Assume $\Gamma_{I_1}(R_1) = \Gamma_{I_2}(R_2) = K_0$. Then $Z_{I_1}(R_1)^* = Z_{I_2}(R_2)^* = \varnothing$. Now, consider $(x_1, y_1), (x_2, y_2) \in Z_{I_1 \times I_2}(R_1 \times R_2)^*$ such that $\text{d}((x_1, y_1), (x_2, y_2)) = 3$. Since $(x_1, y_1) \notin I_1 \times I_2$, assume without loss of generality that $x_1 \in \text{reg}_{I_1}(R_1)$. Then $y_1 \in I_2$. Similarly, $x_2 \in \text{reg}_{I_1}(R_1)$ or $y_2 \in \text{reg}_{I_2}(R_2)$.
  - If $x_2 \in \text{reg}_{I_1}(R_1)$, then $y_2 \in I_2$ because $(x_2, y_2) \in Z_{I_1 \times I_2}(R_1 \times R_2)^*$. Let (a, b) be a vertex adjacent to (x_1, y_1). We know that $x_1 a \in I_1$, so $a \in I_1$. Thus $(a, b)(x_2, y_2) \in I_1 \times I_2$ since $a \in I_1$ and $y_2 \in I_2$. Thus $\text{d}((x_1, y_1), (x_2, y_2)) = 2$.
  - If $y_2 \in \text{reg}_{I_2}(R_2)$, then $x_2 \in I_1$. Thus $(x_1, y_1)(x_2, y_2) \in I_1 \times I_2$, so $\text{d}((x_1, y_1), (x_2, y_2)) = 1$.

  In either case, we have a contradiction.
- Assume $\Gamma_{I_1}(R_1) = K_1$ and $\Gamma_{I_2}(R_2) = K_0$. $I_1 = \{0\}$ by Lemma 3.1. If $R_2 = Z_{I_2}(R_2)$, then $R_2 = I_2$. Let $x \in Z_{I_1}(R_1)^*$. Since $R_2 = Z_{I_2}(R_2)$, the set of vertices is $Z_{I_1 \times I_2}(R_1 \times R_2) = \{(x, i) \mid i \in I_2\}$, all of which are adjacent to each other by definition. So $\Gamma_{I_1 \times I_2}(R_1 \times R_2) = K_{|R_2|}$. This is a contradiction of $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 3$, so $R_2 \neq Z_{I_2}(R_2)$. If $R_1 = Z_{I_1}(R_1)$, then $(x, 0)$ is adjacent to all other vertices in $\Gamma_{I_1 \times I_2}(R_1 \times R_2)$, where $x \in Z_{I_1}(R_1)^*$. Thus $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) \leq 2$. Thus $R_1 \neq Z_{I_1}(R_1)$.
- Assume $\Gamma_{I_1}(R_1) = \Gamma_{I_2}(R_2) = K_1$. Then $I_1 = I_2 = \{0\}$ by Lemma 3.1, so $R_1 \neq Z_{I_1}(R_1)$ and $R_2 \neq Z_{I_2}(R_2)$ by Theorem 2.4.

($\Leftarrow$) Assume $R_1 \neq Z_{I_1}(R_1), R_2 \neq Z_{I_2}(R_2), \Gamma_{I_1}(R_1) = K_1$ and $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) \neq 3$. Let $x_1 \in \text{reg}_{I_1}(R_1), x_2 \in Z_{I_1}(R_1)^*$, and $y \in \text{reg}_{I_2}(R_2)$. Consider the distinct vertices $(x_1, 0)$ and (x_2, y). Clearly, $(x_1, 0)(x_2, y) \notin I_1 \times I_2$ by construction. Thus $\exists (a, b) \in Z_{I_1 \times I_2}(R_1 \times R_2)^*$ such that $(a, b)(x_1, 0) \in I_1 \times I_2$ and $(a, b)(x_2, y) \in I_1 \times I_2$. Since $x_1 a \in I_1$ and $yb \in I_2$, then $(a, b) \in I_1 \times I_2$, which is a contradiction. Thus $\text{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 3$.

(iv) Follows from (i), (ii), (iii).

□

Theorem 3.4. *Let* $\operatorname{diam}(\Gamma_{I_1}(R_1)) = 0$ *and* $\operatorname{diam}(\Gamma_{I_2}(R_2)) = 1$.

(i) $\operatorname{diam}(\Gamma(R_1 \times R_2)) > 0$.
(ii) $\operatorname{diam}(\Gamma(R_1 \times R_2)) = 1$ *if and only if both* $R_1^2 \subseteq I_1$ *and* $R_2^2 \subseteq I_2$ *or* $R_1 = I_1$.
(iii) $\operatorname{diam}(\Gamma(R_1 \times R_2)) = 2$ *if and only if* $R_1 \neq I_1$ *and (without loss of generality)* $R_1^2 \subseteq I_1$ *and* $R_2^2 \nsubseteq I_2$.
(iv) $\operatorname{diam}(\Gamma(R_1 \times R_2)) = 3$ *if and only if* $R_1^2 \nsubseteq I_1$ *and* $R_2^2 \nsubseteq I_2$.

Proof. Let $\operatorname{diam}(\Gamma_{I_1}(R_1)) = 0$ and $\operatorname{diam}(\Gamma_{I_2}(R_2)) = 1$.

(i) Let $x, y \in Z^*_{I_2}(R_2)$ such that $x \neq y$ and $xy \in I_2$. Then $(0, x)(0, y) = (0, xy) \in I_1 \times I_2$, so $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) > 0$.

(ii) (⇒) Let $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 1$.
- Suppose $R_1^2 \nsubseteq I_1$, then $\exists a, b \in R_1$ such that $ab \notin I_1$. Clearly $a, b \notin I_1$. Let $c \in Z^*_{I_2}(R_2)$. Then $(a, c), (b, 0) \in Z^*_{I_1 \times I_2}(R_1 \times R_2)$, but $(a, c)(b, 0) = (ab, 0) \notin I_1 \times I_2$, so $\mathrm{d}((a, c), (b, 0)) > 1$, a contradiction.
- Suppose $R_2^2 \nsubseteq I_2$, then $\exists a, b \in R_2$ such that $ab \notin I_2$. Clearly $a, b \notin I_2$. If $Z^*_{I_1}(R_1) \neq \varnothing$, let $c \in Z^*_{I_1}(R_1)$. Then $(c, a), (0, b) \in Z^*_{I_1 \times I_2}(R_1 \times R_2)$, but $(c, a)(0, b) = (0, ab) \notin I_1 \times I_2$, so $\mathrm{d}((c, a), (0, b)) > 1$, a contradiction. Thus $Z_{I_1}(R_1)^* = \varnothing$, so consider $R_1 - I_1$. If $R_1 - I_1 \neq \varnothing$, then $R_1^2 \nsubseteq I_1$, a contradiction from earlier. Thus $R_1 - I_1 = \varnothing$, so $R_1 = I_1$.

(⇐)
- Suppose $R_1 = I_1$. By Lemma 2.2, $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = \operatorname{diam}(\Gamma_{I_2}(R_2)) = 1$.
- Suppose $R_1^2 \subseteq I_1$ and $R_2^2 \subseteq I_2$. Then for all $(a, b), (c, d) \in Z_{I_1 \times I_2}(R_1 \times R_2)^*$, since $ac \in I_1$ and $bd \in I_2$, $(ac, bd) \in I_1 \times I_2$, so $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 1$.

(iii) (⇒) Let $\operatorname{diam}(\Gamma_{I_1 \times I_2}(R_1 \times R_2)) = 2$.
- By contrapositive of Lemma 2.2, $R_1 \neq I_1$.
- Suppose $Z_{I_1}(R_1)^* \neq \varnothing$. Assume $\mathrm{reg}_{\mathrm{I}_1}(R_1), \mathrm{reg}_{\mathrm{I}_2}(R_2) \neq \varnothing$. Let $a \in \mathrm{reg}_{I_1}(R_1), b \in \mathrm{reg}_{I_2}(R_2), c \in Z_{I_1}(R_1)^*, d \in Z_{I_2}(R_2)^*$. Since a is regular, we need an element from I_1 in the first component to kill a, and since b is regular, we need an element from I_2 in the second component to kill b. Then $\mathrm{ann}_{I_1 \times I_2}((a, d)) = \{(i, l) \mid dl \in I_2$ and $i \in I_1\}$ and $\mathrm{ann}_{I_1 \times I_2}((c, b)) = \{m, i) \mid cm \in I_1$ and $i \in I_2\}$. Since $\mathrm{ann}_{I_1 \times I_2}((a, d)) \cap \mathrm{ann}_{I_1 \times I_2}(c, b) = I_1 \times I_2$, only trivial elements (elements in $I_1 \times I_2$) can kill both (a, d) and (c, b), so $\mathrm{d}((a, d), (c, b)) > 2$, a contradiction. Thus either $\mathrm{reg}_{I_1}(R_1) = \varnothing$ or $\mathrm{reg}_{I_2}(R_2) = \varnothing$.
  - Suppose $\mathrm{reg}_{I_2}(R_2) = \varnothing$. Then $R_2 = Z_{I_2}(R_2)$ and by Lemma 2.3, $R_2^2 \subseteq I_2$. By ii), $R_1^2 \nsubseteq I_1$.
  - Now suppose $\mathrm{reg}_{I_1}(R_1) = \varnothing$. Since $\operatorname{diam}(\Gamma_{I_1}(R_1)) = 0$ and $Z_{I_1}(R_1)^* \neq \varnothing$, $\Gamma_{I_1}(R_1) = K_1$. Then $Z_{I_1}(R_1)^* = \{a\}, a^2 \in I_1$, and $R_1 = \{0, a\}$. Then $R_1^2 \subseteq I_1$, and by ii), $R_2^2 \neq 0$.
- Suppose $Z_{I_1}(R_1)^* = \varnothing$. Then since $R_1 \neq I_1, \mathrm{reg}_{\mathrm{I}_1}(R_1) \neq \varnothing$, and $R_1^2 \nsubseteq I_1$. Assume $\mathrm{reg}_{I_2}(R_2) \neq \varnothing$. Let $a \in \mathrm{reg}_{I_1}(R_1), x \in Z_{I_2}(R_2)^*, r \in \mathrm{reg}_{I_2}(R_2)$. Then $(a, x), (0, r) \in Z_{I_1 \times I_2}(R_1 \times R_2)^*$. Since a is regular, we need an element from I_1 in the first component to kill it a, and since r is regular, we need an element from I_2 in the second component to

kill r. Then $\text{ann}_{I_1\times I_2}((a,x)) = \{(i,y) \mid xy \in I_2 \text{ and } i \in I_1\}$ and $\text{ann}_{I_1\times I_2}((0,r)) = \{(b,i) \mid b \in R_1 \text{ and } i \in I_2\}$. Since $\text{ann}_{I_1\times I_2}((a,x)) \cap \text{ann}_{I_1\times I_2}((0,r)) = I_1 \times I_2$, only trivial elements (elements in $I_1 \times I_2$) can kill both (a,x) and $(0,r)$, so $\text{d}((a,x),(0,r)) > 2$, a contradiction. Thus $\text{reg}_{I_2}(R_2) = \varnothing$, so $R_2 = Z_{I_2}(R_2)$ and by Lemma 2.3, $R_2^2 \subseteq I_2$.

($\Leftarrow$) Assume $R_1^2 \subseteq I_1$, $R_2^2 \nsubseteq I_2$, and $R_1 \neq I_1$. Let $c \in R_1 - I_1$. Then $(c,0)$ is adjacent to every vertex in $\Gamma_{I_1\times I_2}(R_1\times R_2)$, so $\text{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) \leq 2$. Then from i), $\text{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) \neq 1$, so $\text{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) = 2$. Assume $R_1^2 \nsubseteq I_1$ and $R_2^2 \in I_2$. Let $c \in R_2 - I_2$. Then $(0,c)$ is adjacent to every vertex in $\Gamma_{I_1\times I_2}(R_1\times R_2)$, so $\text{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) \leq 2$. Then from i), $\text{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) \neq 1$, so $\text{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) = 2$.

(iv) Follows from (i), (ii), and (iii).

□

Corollary 3.5. *Let* $\text{diam}(\Gamma(R_1)) = 0$ *and* $\text{diam}(\Gamma(R_2)) = 1$.

(i) $\text{diam}(\Gamma(R_1\times R_2)) > 0$.
(ii) $\text{diam}(\Gamma(R_1\times R_2)) = 1$ *if and only if* $R_1^2 = 0 = R_2^2$ *or* $R_1 = \{0\}$.
(iii) $\text{diam}(\Gamma(R_1\times R_2)) = 2$ *if and only if* $R_1 \neq \{0\}$ *and (without loss of generality)* $R_1^2 = 0$ *and* $R_2^2 \neq 0$.
(iv) $\text{diam}(\Gamma(R_1\times R_2)) = 3$ *if and only if* $R_1^2, R_2^2 \neq 0$.

Theorem 3.6. *Let* R_1 *and* R_2 *be rings with ideals* I_1 *and* I_2*, respectively, such that* $\text{diam}(\Gamma_{I_1}(R_1)) = 0$ *and* $\text{diam}(\Gamma_{I_2}(R_2)) = 2$. *Then:*

(i) $\text{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) > 1$.
(ii) $\text{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) = 2$ *if and only if* $R_1 = Z_{I_1}(R_1)$ *or* $R_2 = Z_{I_2}(R_2)$.
(iii) $\text{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) = 3$ *if and only if* $R_1 \neq Z_{I_1}(R_1)$ *and* $R_2 \neq Z_{I_2}(R_2)$.

Proof. (i) Since $\text{diam}(\Gamma_{I_2}(R_2)) \geq 2$, there exist distinct $y_1, y_2 \in Z_{I_2}(R_2)^*$ with $y_1y_2 \notin I_2$. Then $(0,y_1)(0,y_2) = (0,y_1y_2) \notin I_1 \times I_2$. Therefore $\text{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) > 1$.

(ii) ($\Leftarrow$) Let $R_1 = Z_{I_1}(R_1)$ (the proof of Theorem 3.13 satisfies the case where $R_2 = Z_{I_2}(R_2)$). Since $\text{diam}(\Gamma_{I_1}(R_1)) = 0$, either $R_1 = I_1$ or $R_1 = I_1 \cup \{a\}$, where $a^2 \in I_1$. Thus, $R_1^2 \subseteq I_1$. Therefore, $(r,0)(x,y) \in I_1 \times I_2$ for all $r \in R_1$, $(x,y) \in Z_{I_1\times I_2}(R_1\times R_2)^*$. Hence, $\text{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) \leq 2$. By (i), $\text{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) = 2$.

($\Rightarrow$) By the contrapositive of Lemma 2.1.

(iii) By (i) and (ii).

□

Corollary 3.7. *Let* R_1 *and* R_2 *be rings with* $\text{diam}(\Gamma(R_1)) = 0$ *and* $\text{diam}(\Gamma(R_2)) = 2$. *Then:*

(i) $\text{diam}(\Gamma(R_1\times R_2)) > 1$.
(ii) $\text{diam}(\Gamma(R_1\times R_2)) = 2$ *if and only if* $R_1 = Z(R_1)$ *or* $R_2 = Z(R_2)$.
(iii) $\text{diam}(\Gamma(R_1\times R_2)) = 3$ *if and only if* $R_1 \neq Z(R_1)$ *and* $R_2 \neq Z(R_2)$.

Theorem 3.8. *Let* R_1 *and* R_2 *be commutative rings. If* $\text{diam}(\Gamma_{I_1}(R_1)) = 0$ *and* $\text{diam}(\Gamma_{I_2}(R_2)) = 3$*, then:*

(i) $\text{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) > 1$.
(ii) $\text{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) = 2$ *if and only if* $R_1 = Z_I(R_1) \neq I_1$.
(iii) $\text{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) = 3$ *if and only if* $R_1 = I_1$ *or* $R_1 \neq Z_I(R_1)$.

Proof. (i) Same as Theorem 3.6 (i).

(ii) ($\Rightarrow$) Let $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) = 2$. Assume that $R_1 \neq Z_I(R_1)$. Note that since $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_2)(R_2) = 3)$, we can find a minimal path of length 3,

$$y_1 - y_2 - y_3 - y_4$$

We also know there exists $r_1 \in \operatorname{reg}_I(R_1)$. But then consider the path

$$(0, y_1) - (r_1, y_2) - (0, y_3) - (r_1, y_4)$$

Clearly none of the elements in the path are adjacent. If there exists some (i, b) where $i \in I_1$ such that

$$(0, y_1) - (i, b) - (r_1, y_4)$$

is a path, this contradicts the fact that $d(y_1, y_4) = 3$. Thus that path must be minimal as well. But this implies $d((0, y_1),(r_1, y_4)) = 3$, a contradiction. If $Z_I(R_1) = I_1$, then if $j \in I_1$, the path

$$(j, y_1) - (j, y_2) - (j, y_3) - (j, y_4)$$

is clearly minimal, so we get a contradiction here as well.
($\Leftarrow$) Let $R_1 = Z_I(R_1) \neq I_1$. Then there exist $a, b \in R_1$ (not necessarily distinct) where $ab \in I_1$. We know that we have the minimal path in $\Gamma_{I_2}(R_2)$

$$y_1 - y_2 - y_3 - y_4$$

therefore the path $(a, y_1) - (b, y_2) - (0, y_3)$ is minimal, so $\operatorname{diam}\Gamma_{I_1\times I_2}(R_1 \times R_2)) \geq 2$. Note that since $\operatorname{diam}(\Gamma_{I_1}(R_1)) = 0$, if $pq \in I_1$, then $p = q$. Since $R_1 \neq I_1$, we know that p exists. Thus all elements in $\Gamma_{I_1\times I_2}(R_1 \times R_2)$ are of the form (j, r_2) or (p, r_2) for some $r_2 \in R_2$. Thus the element $(p, 0)$ is a bridge between any two other elements, so we know the diameter of $\Gamma(R_1 \times R_2)$ is exactly 2.

(iii) Follows from (i) and (ii).

□

Corollary 3.9. *Let R_1 and R_2 be commutative rings. Then if $\operatorname{diam}(\Gamma(R_1)) = 0$ and $\operatorname{diam}(\Gamma(R_2)) = 3$ then:*

(i) $\operatorname{diam}(\Gamma(R_1 \times R_2)) > 1$.
(ii) $\operatorname{diam}(\Gamma(R_1 \times R_2)) = 2$ *if and only if* $R_1 = Z(R_1) \neq \{0\}$.
(iii) $\operatorname{diam}(\Gamma(R_1 \times R_2)) = 3$ *if and only if* $R_1 = \{0\}$ *or* $R_1 \neq Z(R_1)$.

Theorem 3.10. *Let R_1 and R_2 be commutative rings with ideals I_1 and I_2, respectively, such that $\operatorname{diam}(\Gamma_{I_1}(R_1)) = \operatorname{diam}(\Gamma_{I_2}(R_2)) = 1$. Then*

(i) $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 1$ *if and only if* $R_1^2 \subseteq I_1$ *and* $R_2^2 \subseteq I_2$.
(ii) $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 2$ *if and only if without loss of generality* $R_1^2 \subseteq I_1$ *and* $R_2^2 \nsubseteq I_2$.
(iii) $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 3$ *if and only if* $R_1^2 \nsubseteq I_1$ *and* $R_2^2 \nsubseteq I_2$.

Proof. (i) ($\Rightarrow$) Without loss of generality, let $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 1$ and $R_1^2 \nsubseteq I_1$. Then there exist $x_1, x_2 \in R_1$ such that $x_1x_2 \notin I_1$. Therefore $(x_1, 0)(x_2, 0) \notin I_1 \times I_2$, so $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) > 1$.

($\Leftarrow$) Let $R_1^2 \subseteq I_1$ and $R_2^2 \subseteq I_2$. Then for all $(x_1, y_1),(x_2, y_2) \in R_1 \times R_2$, we know that $x_1x_2 \in I_1$ and $y_1y_2 \in I_2$, so $(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2) \in I_1 \times I_2$.

(ii) ($\Rightarrow$) Assume $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) = 2$. If $R_1^2 \subseteq I_1$ and $R_2^2 \subseteq I_2$, then $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) = 1$ by part (i), which forms a contradiction. Thus assume $R_1^2 \nsubseteq I_1$ and $R_2^2 \nsubseteq I_2$. By Lemma 2.3, there must exist $x_1 \in \operatorname{reg}_{I_1}(R_1)$, and $y_1 \in \operatorname{reg}_{I_2}(R_2)$. Let $x_2 \in Z_{I_1}(R_1)$, $y_2 \in Z_{I_2}(R_2)$ and consider the two elements $(x_1, y_2), (x_2, y_1) \in Z_{I_1\times I_2}(R_1\times R_2)^*$. Clearly $(x_1, y_2)(x_2, y_1) = (x_1x_2, y_1y_2) \notin I_1 \times I_2$ by choice of x_1 and y_1. Since $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) = 2$, there must exist an element $(a, b) \in Z_{I_1\times I_2}(R_1\times R_2)^*$ such that $(x_1, y_2)(a, b) \in I_1\times I_2$ and $(x_2, y_1)(a, b) \in I_1\times I_2$. Thus $x_1a \in I_1$, so $a \in I_1$. Similarly, $y_1b \in I_2$, so $b \in I_2$. Thus $(a, b) \in I_1\times I_2$, so we have a contradiction.

($\Leftarrow$) Assume without loss of generality that $R_1^2 \subseteq I_1$, $R_2^2 \nsubseteq I_2$. Then $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) > 1$ by part (i). Also, since $R_1^2 \subseteq I_1$, for all $x_1, x_2 \in R_1$, $x_1x_2 \in I_1$. Thus $(x_1, 0)$ is adjacent to every vertex in $\Gamma_{I_1\times I_2}(R_1 \times R_2)$, so a path of length 2 can be found between any two vertices in $\Gamma_{I_1\times I_2}(R_1 \times R_2)$ by way of the vertex $(x_1, 0)$. Thus $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 2$.

(iii) Follows directly from parts (i) and (ii).

□

Theorem 3.11. *Let R_1 and R_2 be commutative rings with ideals I_1 and I_2, respectively, such that* $\operatorname{diam}(\Gamma_{I_1}(R_1)) = 1$ *and* $\operatorname{diam}(\Gamma_{I_2}(R_2)) = 2$. *Then:*

(i) $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) > 1$.
(ii) $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 2$ *if and only if* $R_1 = Z_{I_1}(R_1)$ *or* $R_2 = Z_{I_2}(R_2)$.
(iii) $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 3$ *if and only if* $R_1 \neq Z_{I_1}(R_1)$ *and* $R_2 \neq Z_{I_2}(R_2)$.

Proof. (i) Same as Theorem 3.6 (i).

(ii) ($\Leftarrow$) Let $R_1 = Z_{I_1}(R_1)$ (the case where $R_2 = Z_{I_2}(R_2)$ is addressed by the proof of Theorem 3.13). Thus, we have $R_1^2 \subseteq I_1$ by Lemma 2.3. Let $a \in R_1^*$. Since $(a, 0)(x, y) \in I_1 \times I_2$ for all $(x, y) \in Z^{(}_{I_1\times I_2}R_1 \times R_2)^*$, we have $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) \leq 2$. It follows from (i) that $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 2$.

($\Rightarrow$) Assume that $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 2$, $R_1 \neq Z_{I_1}(R_1)$, and $R_2 \neq Z_{I_2}(R_2)$. Let $x \in Z_{I_1}(R_1)^*$, $y \in Z_{I_2}(R_2)^*$, $m \in \operatorname{reg}_{I_1}(R_1)$, $n \in \operatorname{reg}_{I_2}(R_2)$. Then $(x, n)(m, y) \notin I_1 \times I_2$. Since $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 2$, there exists $(a, b) \in Z_{I_1\times I_2}(R_1 \times R_2)^*$ such that $(x, n)(a, b), (m, y)(a, b) \in I_1 \times I_2$. Then $ma \in I_1$ and $nb \in I_2$, so $(a, b) \in I_1 \times I_2$, a contradiction.

(iii) Follows from (i) and (ii).

□

Theorem 3.12. *Let R_1 and R_2 be commutative rings with ideals I_1 and I_2, respectively, such that* $\operatorname{diam}(\Gamma_{I_1}(R_1)) = 1$ *and* $\operatorname{diam}(\Gamma_{I_2}(R_2)) = 3$. *Then:*

(i) $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) > 1$.
(ii) $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 2$ *if and only if* $R_1 = Z_{I_1}(R_1)$.
(iii) $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 3$ *if and only if* $R_1 \neq Z_{I_1}(R_1)$.

Proof. (i) Same as Theorem 3.6 (i).

(ii) ($\Leftarrow$) Same as Theorem 3.11 (ii).

($\Rightarrow$) Assume that $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 2$ and $R_1 \neq Z_{I_1}(R_1)$. Let $m \in \operatorname{reg}_{I_1}(R_1)$. Since $\operatorname{diam}(\Gamma_{I_2}(R_2)) = 3$, there exist distinct $y_1, y_2 \in Z_{I_2}(R_2)^*$ with $y_1y_2 \notin I_2$, and there is no $y_3 \in Z_{I_2}(R_2)^*$ such that $y_1y_3, y_2y_3 \in I_2$. Now $(m, y_1), (m, y_2) \in Z_{I_1\times I_2}(R_1 \times R_2)^*$, and $(m, y_1)(m, y_2) \notin I_1 \times I_2$.

Since $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 2$, there exists $(a, b) \in Z_{I_1\times I_2}(R_1 \times R_2)^*$ such that $(m, y_1)(a, b), (m, y_2)(a, b) \in I_1 \times I_2$. Then $ma \in I_1$, so we have $a \in I_1$. Also, $y_1b, y_2b \in I_2$. Hence, $b \in I_2$. Thus, $(a, b) \in I_1 \times I_2$, a contradiction.

(iii) By (i) and (ii).

□

Theorem 3.13. *Let R_1 and R_2 be commutative rings with ideals I_1 and I_2, respectively, such that* $\operatorname{diam}(\Gamma_I(R_1)) = \operatorname{diam}(\Gamma_I(R_2)) = 2$. *Then:*

(i) $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) > 1$
(ii) $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 2$ *if and only if* $R_1 = Z_{I_1}(R_1)$ *or* $R_2 = Z_{I_2}(R_2)$
(iii) $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 3$ *if and only if* $R_1 \neq Z_{I_1}(R_1)$ *and* $R_2 \neq Z_{I_2}(R_2)$

Proof. (i) Same as Theorem 3.6 (i).

(ii) (⇐) Without loss of generality, let $R_1 = Z_{I_1}(R_1)$. Since $R_1 = Z_{I_1}(R_1)$, by Lemma 3.2, for all $x_1, x_2 \in Z_{I_1}(R_1)$, there exists $x_3 \in R_1 - I_1$ such that $x_3x_1, x_3x_2 \in I_1$. So for all $(x_1, y_1), (x_2, y_2) \in Z_{I_1\times I_2}(R_1\times R_2)^*$, there exists $(x_3, 0) \in Z_{I_1\times I_2}(R_1 \times R_2)^*$ such that $(x_1, y_1)(x_3, 0) \in I_1 \times I_2$.

If without loss of generality $(x_2, y_2) = (x_3, 0)$ then $(x_1, y_1)(x_2, y_2) \in I_1 \times I_2$. Thus $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) \leq 2$ and by (i), $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1\times R_2)) = 2$.

(⇒) Same as Theorem 3.11 (ii).

(iii) By (i) and (ii).

□

Theorem 3.14. *Let R_1 and R_2 be commutative rings with ideals I_1 and I_2, respectively, such that* $\operatorname{diam}(\Gamma_{I_1}(R_1)) = 2$ *and* $\operatorname{diam}(\Gamma_{I_2}(R_2)) = 3$. *Then:*

(i) $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) > 1$.
(ii) $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 2$ *if and only if* $R_1 = Z_{I_1}(R_1)$.
(iii) $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 3$ *if and only if* $R_1 \neq Z_{I_1}(R_1)$.

Proof. (i) Same as Theorem 3.6 (i).

(ii) (⇐) Same as Theorem 3.13 (ii).

(⇒) Let $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 2$. Assume that $R_1 \neq Z_{I_1}(R_1)$. Let $k \in \operatorname{reg}_{I_1}(R_1)$. Since $\operatorname{diam}(\Gamma_{I_2}(R_2)) = 3$, we know that we can find a minimal path

$$y_1 - y_2 - y_3 - y_4$$

where $\mathrm{d}(y_1, y_4) = 3$. Similarly, we know that that since $\Gamma_{I_1}(R_1)$ has diameter 2, we can find x_1 and x_2 such that x_1 is adjacent to x_2. Then consider the following path in $\Gamma_{I_1\times I_2}(R_1 \times R_2)$:

$$(k, y_1) - (0, y_2) - (x_1, y_3) - (x_2, y_4)$$

Note that (k, y_1) can't be adjacent to (x_2, y_4) since $k \in \operatorname{reg}_{I_1}(R_1)$. Then there must exist some (a, b) such that $(k, y_1)(a, b) \in I_1\times I_2$ and $(a, b)(x_2, y_4) \in I_1 \times I_2$. Note that this forces $a \in I_1$. Then b can't be in I_2. But then $d(y_1, y_4) = 2$, which is a contradiction.

Thus $R_1 = Z_{I_1}(R_1)$.

(iii) By (i) and (ii).

□

Theorem 3.15. *Let R_1 and R_2 be commutative rings with ideals I_1 and I_2, respectively, such that* $\operatorname{diam}(\Gamma_{I_1}(R_1)) = \operatorname{diam}(\Gamma_{I_2}(R_2)) = 3$. *Then* $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 3$.

Proof. Since $\operatorname{diam}(\Gamma_{I_1}(R_1)) = 3$, there exist vertices $x_1, x_2, x_3, x_4 \in \Gamma_{I_1}(R_1)$ such that there is a minimal path

$$x_1 - x_2 - x_3 - x_4$$

and $\mathrm{d}(x_1, x_4) = 3$. Similarly, we can find vertices $y_1, y_2, y_3, y_4 \in \Gamma_{I_2}(R_2)$ such that there is a minimal path

$$y_1 - y_2 - y_3 - y_4$$

and $d(y_1, y_4) = 3$.

Now consider the following path in $\Gamma_{I_1\times I_2}(R_1 \times R_2)$;

$$(x_1, y_1) - (x_2, y_2) - (x_3, y_3) - (x_4, y_4)$$

Assume that $d((x_1, y_1), (x_4, y_4)) < 3$. Then there are two cases:

- (x_1, y_1) is adjacent to (x_4, y_4). But this would imply that x_1 was adjacent to x_4 in $\Gamma_{I_1}(R_1)$, which would contradict the fact that $\mathrm{d}(x_1, x_4) = 3$.
- (x_1, y_1) is adjacent to some (a, b) which is adjacent to (x_4, y_4). Assume $a \in I_1$. But then $y_1 b \in I_2$ and $by_4 \in I_2$ which would imply that in $\Gamma_{I_2}(R_2)$, we would have the path

$$y_1 - b - y_4$$

  which contradicts $\mathrm{d}(y_1, y_4) = 3$. We get a similar contradiction if $b \in I_2$.

Thus $\mathrm{d}((x_1, y_1), (x_4, y_4)) \geq 3$. Since the diameter of a ideal-divisor graph is always bounded by 3, we get that $\operatorname{diam}(\Gamma_{I_1\times I_2}(R_1 \times R_2)) = 3$. □

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