Commutative Rings with Domain-type Properties

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Throughout, R will be a commutative ring with identity with total quotient ring $T(R)$, group of units $U(R)$, set of zero-divisors $Z(R)$, and Jacobson radical $J(R)$. For $a, b \in R$, we define three associate relations:

1. We say a and b are *associate*, denoted $a \sim b$, if $a|b$ and $b|a \Leftrightarrow (a) = (b)$.
2. We say a and b are *strongly associate*, denoted $a \approx b$, if there exists a $u \in U(R)$ such that $a = ub$.
3. We say a and b are *very strongly associate*, denoted $a \cong b$, if $a \sim b$ and either $a = 0$ or $a = rb \Rightarrow r \in U(R)$.

Definition 1 *A ring R is called présimplifiable if $xy = x$ for $x,\ y \in R$ implies that either $x = 0$ or $y \in U(R)$.*

We have the following theorem:

Theorem 2 *For a commutative ring R the following are equivalent:*

1. $a \sim b \Rightarrow a \cong b$ *for all* $a, b \in R$,
2. $a \approx b \Rightarrow a \cong b$ *for all* $a, b \in R$,
3. R *is présimplifiable, and*
4. $Z(R) \subseteq J(R)$.

Proof. (1) $\Rightarrow$ (2). Clear

(2) $\Rightarrow$ (3). Suppose $xy = x$ and $x \neq 0$. Since $x = 1 \cdot x$, we have $x \approx x$. From (2) we get $x \cong x$, and hence $y \in U(R)$.

(3) $\Rightarrow$ (4). Let $x \in Z(R)$ with $xy = 0$ and $y \neq 0$. It suffices to show $1 - xz \in U(R)$ for all $z \in R$. So, $xy = 0 \Rightarrow -xzy = 0 \Rightarrow -xzy + y = y \Rightarrow y(1 - xz) = y \Rightarrow 1 - xz \in U(R)$ by (3).

(4) $\Rightarrow$ (3). Let $xy = x$ and $x \neq 0$. Thus, $x(1-y) = 0 \Rightarrow 1-y \in Z(R) \subseteq J(R)$. So, $1 - (1 - y) = y \in U(R)$.

(3) $\Rightarrow$ (1). Suppose $a \sim b$. Then, $a = xb$ and $b = ya$. So, $a = xya \Rightarrow xy \in U(R) \Rightarrow x \in U(R)$. ■

It is easy to check that $\mathbb{Z}_n$ is présimplifiable if and only if $n = p^m$ where p is some prime.

Definition 3 *A ring R is an* ***associate ring*** *if $a \sim b \Rightarrow a \approx b$. A ring R is a* ***superassociate ring*** *if every subring of R is an associate ring.*

If R is présimplifiable, R is an associate ring, since $a \sim b \Rightarrow a = rb$ and $b = sa \Rightarrow a = rsa \Rightarrow a = 0$ or $rs \in U(R) \Rightarrow r \in U(R)$. The converse is false, since a direct product of associate rings is associate (see below), but a présimplifiable ring has no nontrivial idempotents and hence is indecomposable. Also, any integral domain or any quasi-local ring is présimplifiable and hence an associate ring. We see a quasi-local ring is présimplifiable by the following:

Let R be a quasi-local ring with maximal ideal M and suppose $xy = x$. Then, $x(1 - y) = 0 \in M \Rightarrow x \in M$ or $1 - y \in M$. If $x \notin M$, then $x \in U(R) \Rightarrow 1 - y = 0 \Rightarrow y \in U(R)$. If $x \in M$, then either $1 - y \in M \Rightarrow y \in U(R)$, or $1 - y \in U(R) \Rightarrow x = 0$.

Note that if $a \in R$ is regular, then $a \sim b \Rightarrow a \cong b \Rightarrow a \approx b$, since $a = rsa \Rightarrow a(1 - rs) = 0 \Rightarrow 1 - rs = 0 \Rightarrow r \in U(R)$.

Theorem 4 *Let $\{R_\alpha\}$ be a nonempty family of commutative rings. Then, $\prod R_\alpha$ is an associate ring $\Leftrightarrow$ each R_α is an associate ring.*

Proof. Let $a = (a_\alpha), b = (b_\alpha) \in \prod R_\alpha$. Then, $a \sim b \Leftrightarrow$ each $a_\alpha \sim b_\alpha$ and $a \approx b \Leftrightarrow$ each $a_\alpha \approx b_\alpha$. The result follows. ■

Thus, any PIR is an associate ring, since a PIR decomposes into domains and SPIR's, and SPIR's are associate since they are local. Also, any zero-dimensional Noetherian ring is associate, since zero-dimensional Noetherian implies Artinian and is thus isomorphic to a finite direct product of Artinian local rings.

However, the class of associate rings is not closed under homomorphic images, subrings, or subdirect products.

Example 5 *For any ring R, $R[x,y,z]/(x-xyz)$ is not an associate ring. To see this, first note that $\bar{x} \sim \bar{x}\bar{y}$ in $R[x,y,z]/(x-xyz)$. Now suppose $\bar{f}\bar{x} = \bar{x}\bar{y}$. for $f \in R[x,y,z]$. We must show that $\bar{f}$ can't be a unit. Now, $\bar{f}\bar{x}-\bar{x}\bar{y} = \bar{0}$, so $fx-fy \in (x-xyz)$. Therefore $fx-fy = xh(1-yz)$ for some $h \in R[x,y,z]$. Then $x(f-y-h(1-yz)) = 0$, which implies $f-y-h(1-yz) = 0$, so $f = y+h(1-yz)$. If f is a unit, then $(f,x) = R[x,y,z]$. But, if $y = z$ and $x = 0$ we get $(z+h(1-z^2)) = R[z]$, which is false. Therefore, $\bar{f}$ cannot be a unit, so $R[x,y,z]/(x-xyz)$ is not associate. Thus any ring R can be embedded in a nonassociate ring.*

Note that if K is a field, then the integral domain $K[x,y,z]$ is an associate ring (in fact, it is superassociate), while $K[x,y,z]/(x-xyz)$ is not an associate ring. Thus, homomorphic images of associate rings are not necessarily associate. We see that $\overline{x} \sim \overline{xy}$, since clearly $(\overline{xy}) \subset (\overline{x})$, and $(\overline{x}) \subset (\overline{xy})$ since $\overline{x} = \overline{xyz}$. Also, since a unit of $K[x,y,z]/(x-xyz)$ is some $\overline{f}$ such that $fg-1 \in (x-xyz)$ for some $g \in K[x,y,z]$, we have $fg-1 = h \in (x-xyz) \Rightarrow fg-h = 1$. So, $\overline{x} \not\approx \overline{xy}$ since $\overline{z}$ is not a unit (no constant term).

Note that since $(x-xyz) = (x)(1-yz) = (x) \cap (1-yz)$, $K[x,y,z]/(x-xyz)$ is a subdirect product of the two integral domains $K[x,y,z]/(x) \cong K[y,z]$ and $K[x,y,z]/(1-yz) \cong K[x,y,y^{-1}]$, since

$$\begin{aligned} K[x,y,z]/(x-xyz) &\xrightarrow{\pi_1} K[x,y,z]/(x-xyz) \Big/ (x-xyz)/(x) \\ &\cong K[x,y,z]/(x) \end{aligned}$$

$$\begin{aligned} K[x,y,z]/(x-xyz) &\xrightarrow{\pi_2} K[x,y,z]/(x-xyz) \Big/ (x-xyz)/(1-yz) \\ &\cong K[x,y,z]/(1-yz) \end{aligned}$$

So, we see the class of associate rings is not closed under subdirect products and hence not closed under subrings.

Actually, any reduced ring is a subdirect product of associate rings $R \hookrightarrow \prod \{R/P \mid P \text{ is a minimal prime of } R\}$. Thus, if R is reduced with a finite number of minimal primes, R is a finite subdirect product of associate rings. Also, for any R, $R \hookrightarrow \prod_{M \in \mathrm{Max}(R)} R_M$. So, every ring is a subring of a direct product of associate rings. On the other hand, $K \subset K[x,y,z]/(x - xyz)$, where K is associate, and $K[x,y,z]/(x - xyz)$ is not associate.

Using Theorem 4 it is easy to see that $\mathbb{Z}_n$ is associate for every $n \in \mathbb{N}$.

Lemma 6 *If a and b are both idempotents in a ring R, then $a \sim b \Rightarrow a \approx b$.*

Proof. Let M be a maximal ideal of R. Then, $\frac{a}{1}$ and $\frac{b}{1}$ are idempotents in R_M with $R_M a = R_M b$. Now since R_M is quasi-local, R_M has no nontrivial idempotents. Hence, since $R_M a = R_M b$ and $\frac{a}{1}, \frac{b}{1}$ are idempotent, we have $\frac{a}{1} = \frac{0}{1} = \frac{b}{1}$ or $\frac{a}{1} = \frac{1}{1} = \frac{b}{1}$. Thus, $\frac{a-b}{1} = 0$ in every R_M. Hence, $a - b = 0 \Rightarrow a = b$. ■

Recall that R is a von Neumann regular ring if every element is von Neumann regular, i.e. for each $a \in R$, there exists an $x \in R$ such that $axa = a$. Thus, a von Neumann regular ring is présimplifiable if and only if it is a field.

Lemma 7 *If R is von Neumann regular and $a \in R$, then there exist $u \in U(R)$ and an idempotent $e \in R$ such that $a = ue$.*

Proof. Let $a \in R$. Then there exists an $x \in R$ such that $a = axa$. Then $(ax)^2 = axax = (axa)x = ax$, and ax is idempotent. Thus, $a = ax[a + (1 - ax)] = e[a + (1 - e)]$.

We claim $a + (1 - e)$ is a unit. It suffices to show $a + (1 - e)$ is a unit for any localization. Since any localization of a von Neumann regular ring is a field, we need only show $a + (1 - e)$ is nonzero in every localization.

Case 1: $\frac{a}{1} = \frac{0}{1} \Rightarrow \frac{e}{1} = \frac{0}{1}$ since $e = ax \Rightarrow \frac{a+(1-e)}{1} = \frac{1}{1}$

Case 2: $\frac{a}{1} \neq \frac{0}{1} \Rightarrow \frac{e}{1} \neq \frac{0}{1}$ since $(e) = (a) \Rightarrow R_M = R_M a = R_M e \Rightarrow \frac{e}{1} = \frac{1}{1} \Rightarrow \frac{1-e}{1} = \frac{0}{1} \Rightarrow \frac{a+(1-e)}{1} = \frac{a}{1} \neq \frac{0}{1}$.

So, $a + (1 - e)$ is locally and hence globally a unit, and $a = eu$, where $e = ax$ is an idempotent and $u = a + (1 - e)$ is a unit. ■

Theorem 8 *A von Neumann regular ring R is associate.*

Proof. Let $a, b \in R$ with $a \sim b$. By Lemma 7 we can write $a = u_1 e_1$ and $b = u_2 e_2$ where $u_1, u_2 \in U(R)$ and $e_1, e_2 \in R$ are idempotent. Then, $e_1 \sim a \sim b \sim e_2 \Rightarrow e_1 = e_2$. By Lemma 6, we get $a \approx b$. ■

Definition 9 *R is **domainlike** if $Z(R) \subseteq \operatorname{nil}(R)$, the nilradical of R.*

Lemma 10 *(0) is primary $\Leftrightarrow$ R is domainlike.*

Proof. ($\Rightarrow$) Let (0) be primary. Then, if $ab = 0$ and $a \neq 0$, $b^n = 0$ for some n. Let $b \in Z(R)$. Then there exists an $a \in R - \{0\}$ such that $ab = 0$. Hence, $b^n = 0$, and $b \in \operatorname{nil}(R)$.

($\Leftarrow$) Let R be domainlike. Suppose $ab = 0$ and $a \neq 0$. Then, $b \in Z(R) \subseteq \operatorname{nil}(R)$, which implies $b^n = 0$ for some n. Hence, (0) is primary. ■

Fact: If R is domainlike, then R is présimplifiable, since $\operatorname{nil}(R) \subseteq J(R)$ (see Theorem 2).

Example 11 *The converse of the previous statement is false. Let $R = K[[x,y]]/(x)(x,y)$. Then, R is local since $K[[x,y]]$ is local, and the homomorphic image of a local ring is local. Hence, R is présimplifiable. However, R is not domainlike since $Z(R) = (\overline{x}, \overline{y})$ while $\operatorname{nil}(R) = (\overline{x})$.*

Note: We have previously shown that

R quasi-local $\Rightarrow$ R is présimplifiable $\Rightarrow$ R is associate, and that

R domainlike $\Rightarrow$ R is présimplifiable $\Rightarrow$ R is associate.

However there is no strong implication between domainlike and quasi-local. Example 11 shows that a quasilocal (in fact, local) ring need not be domainlike, hence a quasilocal ring need not be domainlike. Further $\mathbb{Z}$ is domainlike, présimplifiable, and associate, but not quasi-local. It is also of interest to note that a domainlike ring can be neither Noetherian nor quasilocal. For example, $R = \mathbb{Z}[2X, 2X^2, 2X^3, ...]$ is domainlike (a subring of $\mathbb{Z}[X]$) and R is not Noetherian (Hutchins) since the ideal $P = (2X, 2X^2, 2X^3, ...)$ cannot be finitely generated. Further, from Hutchins we have that $(2, 2X, 2X^2, 2X^3, ...)$ is maximal. Then, since $(3, 2X, 2X^2, 2X^3, ...) \subsetneq (2, 2X, 2X^2, 2X^3, ...)$ we get that $(3, 2X, 2X^2, 2X^3, ...)$ is also maximal in R - thus R is domainlike, not quasilocal, and not Noetherian.

Remark 12 *Any subring of a domainlike ring is again domainlike, but a subring of a présimplifiable ring need not be présimplifiable. So, R domainlike implies that R is superassociate. The converse is false [Principal Ideals and Associate Rings, Remark 3]. Indeed, R is présimplifiable if and only if $R[[x]]$ is présimplifiable (DDAI, pg. 471), but $R[x]$ is présimplifiable if and only if (0) is primary (which implies that R is présimplifiable) [DDAI, p. 472].*

Proof. ($\Rightarrow$) Let $a, b \in R$ such that $a \sim b$ in R. Thus, $a \sim b$ in $R[x]$ and $a \cong b$ in $R[x]$. Then, $a \cdot f = b$, where $f \in U(R[x])$. Thus, f must have a constant coefficient $c \in U(R)$. It follows that $ac = b$, and $a \cong b$ in R.

($\Leftarrow$) $Z(R[x]) = \{f \mid \exists r \in R \text{ with } rf = 0\}$. Let $f = a_0 + a_1 x + \cdots a_n x^n \in Z(R[x])$. Then, $ra_i = 0$ for all i. So, since $r \neq 0$, there is an n_i so that $a_i^{n_i} = 0$ for all i. So, each a_i is nilpotent, which implies f is nilpotent, and $f \in \operatorname{nil}(R[x])$. Since, $\operatorname{nil}(R[x]) = J(R[x])$, we have $f \in J(R[x])$. ■

This also gives

Lemma 13 *$R[x]$ is présimplifiable $\Leftrightarrow$ $R[x]$ is domainlike $\Leftrightarrow$ R is domainlike. (See Remark 32).*

Note that $\mathbb{Z}_m[x]$ is présimplifiable iff $m = p^n$, since $\mathbb{Z}_m$ is domainlike iff $m = p^n$. Now, since a domainlike ring is présimplifiable, a domainlike ring is associate. Thus, any domainlike ring is superassociate. Since an integral domain is superassociate, Example 5 shows that the homomorphic image of a subdirect product or direct product of superassociate rings need not be associate, let alone superassociate.

Also, note that every subring of $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ is an associate ring, but $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are not domainlike.

Proposition 14 *If R is domainlike, so is $T(R)$.*

Proof. Assume R is domainlike. We will show (0) is primary in $T(R)$. Assume $\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{0}{1}$ and $\frac{r_1}{s_1} \neq \frac{0}{1}$. So, there exists an $\overline{s} \in \operatorname{reg}(R)$ such that $\overline{s}(r_1 r_2 - 0) = 0$. Thus, $r_1 r_2 = 0$. If $r_1 \neq 0$, then $r_2^n = 0$ for some n, since (0) is primary in R. Hence, $\left(\frac{r_2}{s_2}\right)^n = \frac{0}{1}$, and so (0) is primary in $T(R)$. ■

Definition 15 *Let R be a commutative ring with identity and let M be an R-module. The idealization of M in R, denoted $R(M)$, is the set $\{(r, m) \mid r \in R, m \in M\}$ with $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$ and $(r_1, m_1) \cdot (r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$.*

Example 16 *R présimplifiable $\not\Rightarrow$ $R[x]$ is associate. For such an example, consider $R = \mathbb{Z}_{(2)}(\mathbb{Z}_4)$ (Ex. 6.1 on page 472 of DDAI).*

Therefore, R associate does not imply that $R[x]$ is associate, but $R[x]$ associate does imply that R is associate (if $a \sim b$ in R then $a \sim b$ in $R[x]$. Thus, $au = b$ where $u = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n$ with $a_0 \in U(R)$ - thus $u \in U(R[x])$. This will imply that $aa_0 = b$.)

Definition 17 *Let M be an R-module. Let $m_1, m_2 \in M$. Define $m_1 \sim m_2 \Leftrightarrow Rm_1 = Rm_2$; $m_1 \approx m_2 \Leftrightarrow m_2 = um_1$ for some $u \in U(R)$; $m_1 \cong m_2 \Leftrightarrow m_1 \sim m_2$ and $m_1 = rm_2$ implies $r \in U(R)$. Call M an **associate R-module** if $m_1 \sim m_2 \Rightarrow m_1 \approx m_2$ and call M a **présimplifiable R-module** if $m_1 \sim m_2 \Rightarrow m_1 \cong m_2$.*

Note: An R-module M is associate (présimplifiable) if and only if each cyclic R-module is associate (présimplifiable). Therefore, a submodule of an associate (présimplifiable) module is associate (présimplifiable) since a submodule is the union of the cyclic R-modules generated by its constituent elements. In other words, for a module, associate implies superassociate.

Proposition 18 *If $R(M)$ is associate, then R is associate and M is an associate R-module.*

Proof. Suppose R is not associate. Then, there are $r_1, r_2 \in R$ such that $r_1 \sim r_2$, but $r_1 \not\approx r_2$. So, $(r_1, 0) \sim (r_2, 0)$, but $(r_1, 0) \not\approx (r_2, 0)$, since any unit of $R(M)$ is of the form (u, m) where $u \in U(R)$, a contradiction.

Suppose M is not an associate R-module. Then, there are $m_1, m_2 \in M$ such that $Rm_1 = Rm_2$, but there is not a $u \in U(R)$ so that $m_1 = um_2$. So, clearly $(0, m_1) \sim (0, m_2)$ and if $(0, m_1) \approx (0, m_2)$, then there exists $(u, \overline{m})$ with $u \in U(R)$ such that $(0, m_1) = (0, m_2)(u, \overline{m}) \Rightarrow (0, m_1) = (0, um_2)$, a contradiction. ∎

Theorem 19 *Let R be présimplifiable and M be an R-module.*

1. *$R(M)$ is associate if and only if M is associate,*
2. *$R(M)$ is présimplifiable if and only if M is présimplifiable.*

Proof. (2) In DDAII, p. 209, prop 3.1

(1) ($\Rightarrow$) If $m_1 \sim m_2$ in M, then $(0, m_1) \sim (0, m_2)$ in $R(M)$. So, $(0, m_2) = (u, n)(0, m_1)$, where $(u, n) \in U(R(M))$. Hence, $u \in U(R)$, and $m_1 \approx m_2$. So, M is associate.

($\Leftarrow$) If $(0, m_1) \sim (0, m_2)$ in $R(M)$, then $m_1 \sim m_2$ in M, and so $m_2 = um_1$ for some $u \in U(R)$. So, $(0, m_2) = (u, 0)(0, m_1)$, and $(0, m_2) \approx (0, m_1)$. Now, suppose $(a, m_1) \sim (b, m_2)$ where $a \neq 0$ (which implies $b \neq 0$). Then, $(a, m_1) = (c, n)(b, m_2)$ implies $a = cb$. Hence, since R is présimplifiable and $(a) = (b)$, $c \in U(R)$. This implies $(c, n) \in U(R(M))$, and $(a, m_1) \approx (a, m_2)$.
∎

Proposition 20 *A cyclic abelian group A (as a $\mathbb{Z}$-module) is associate if and only if $A \cong \mathbb{Z}$, or $A \cong \mathbb{Z}_n$ for $n = 1, 2, 3, 4, 6$. However, if A is présimplifiable, then $A \cong \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_3,$ or $\{0\}$.*

Proof. The second statement is obvious. Also, it is easy to check that $\mathbb{Z}$ and $\mathbb{Z}_n$ are associate for $n = 1, 2, 3, 4, 6$. Conversely, assume A is associate. Clearly, $\mathbb{Z}$ is associate. Let $A = \mathbb{Z}_n$, where $\mathbb{Z}_n$ is associate. If $1 \leq l \leq n-1$, with $(l, n) = 1$, then $l \sim 1$ (since $ls + tn = 1 \Rightarrow l \cdot s = 1 \mod n$). Hence, $l = \pm 1 \mod n$, since $U(\mathbb{Z}) = \pm 1$. So, either $l = 1$ or $l = n - 1$. Hence, $\phi(n) = 1$ or $\phi(n) = 2$. So, $n = 2, 3, 4$ ■

Example 21 *An associate ring need not be expressible as a direct product of présimplifiable rings. For example, $\mathbb{Z}(\mathbb{Z}_4)$ is associate by Theorem 19 and Proposition 20. However, $\mathbb{Z}(\mathbb{Z}_4)$ has no nontrivial idempotents ((0,0) and (1,0) are the only idempotents). Therefore $\mathbb{Z}(\mathbb{Z}_4)$ cannot be written as a direct product of présimplifiable rings. Note that a présimplifiable ring has no nontrivial idempotents and is hence indecomposable, but this example shows that the converse is false - $\mathbb{Z}(\mathbb{Z}_4)$ is an indecomposable associate ring that is not présimplifiable.*

Corollary 22 *As a $\mathbb{Z}$-module, an abelian group G is associate if and only if $G = F \oplus T$, where F is torsion-free and T is torsion with $3T = 0$, $4T = 0$, or $6T = 0$.*

Proof. ($\Leftarrow$) By hypothesis, each element of G has order $\infty, 2, 3, 4$, or 6, so the result follows.

($\Rightarrow$) The torsion part T is of bounded order, so $G = F \oplus T$. Each element of infinite order is isomorphic to $\mathbb{Z}$. If $a \in T$ has finite order, $4a = 0$, $3a = 0$, or $6a = 0$. So, $\mathbb{Z}a$ is associate. ■

Remark 23 *Suppose p is prime. Then every ideal of $R = \mathbb{Z} \oplus \mathbb{Z}_p$ is generated by two elements.*

Proof. Suppose $0 \neq I \subsetneq R$ is an ideal of R. If $(0, a) \in I$ where $a \neq 0$, then $(0, 1) \in I$. Thus, $I / (0, 1)$ is principal. So, assume no $(0, a) \in I$ where $a \neq 0$. Now, some $(n, 0) \in I$, since $(m, a) \in I \Rightarrow (pm, pa) = (pm, 0) \in I$. Choose n_1 to be the least positive integer with $(n_1, 0) \in I$. Then, $(n, 0) \in I \Rightarrow (n, 0) \in \langle (n_1, 0) \rangle$. If $I = \langle (n_1, 0) \rangle$, we are done. So, assume some $(n, a) \in I$ with $n \neq 0$, $a \neq 0$. Then, some $(m, 1) \in I$ as before. Let n_2 be the least positive integer such that $(n_2, 1) \in I$. We claim that $I = \langle (n_1, 0), (n_2, 1) \rangle$. If $(n, 1) \in I$, then $(n_2, 1) - (n, 1) = (n_2 - n, 0) \in \langle (n_1, 0) \rangle$. If $2 \leq a \leq p - 1$, $(n, a) - a(n_2, 1) = (n - an_2, 0) \in \langle (n_1, 0) \rangle$. So, $(n, a) \in \langle (n_1, 0), (n_2, 1) \rangle$. ■

Example 24 *$R = \mathbb{Z}(\mathbb{Z}_5)$ is not associate (as a $\mathbb{Z}$-module), but every ideal of R is generated by 2 elements. So, even though a PIR is associate, if every ideal of R is generated by two elements then R need not be associate.*

Definition 25 *An R-module, M, **preserves** $Z(R)$ if $rm = 0$ in M where $m \neq 0$ and $r \neq 0$ implies that $r \in Z(R)$.*

Theorem 26 *$R(M)$ is domainlike $\Leftrightarrow$ R is domainlike and M preserves $Z(R)$.*

Proof. $(\Rightarrow)$ Assume that $R(M)$ is domainlike, so $((0,0))$ is primary in $R(M)$. Let $ab = 0$ in R and let $a \neq 0$. Thus $(a,0)(b,0) = (0,0)$ in $R(M)$ and $(a,0) \neq (0,0)$. Since $R(M)$ is domainlike, we have that $(b,o)^n = (0,0)$ for some n. So, $(b^n, 0) = (0,0) \Rightarrow b^n = 0$ in R. Thus (0) is primary in R and hence R is domainlike. Now, assume that for some $m \neq 0$ in M and some $r \neq 0$ in R we have that $rm = 0$ in M. Therefore $(r, 0)(0, m) = (0,0)$ in $R(M)$ and $(0, m) \neq (0,0)$. $R(M)$ domainlike implies that $(r, 0)^n = (0,0)$ for some n. So $r^n = 0$ in R and hence $r \in Z(R)$.

$(\Leftarrow)$ Let $(a, l)(b, m) = (ab, am + bl) = (0,0)$ in $R(M)$ and $(a, l) \neq (0,0)$. If $a \neq 0$ in R, then R domainlike implies that $b^n = 0$ in R for some n. Thus $(b, m)^{2n} = (b, m)^n (b, m)^n = (0, k)(0, k) = (0, 0)$. If $a = 0$ in R, then $l \neq 0$ in M and $bl = 0$ in M. So $b \in Z(R)$ by hypothesis. Thus b is nilpotent since R is domainlike, so $b^n = 0$ in R for some n. Thus, $(b, m)^{2n} = (0,0)$ as before. So $R(M)$ is domainlike. ■

Remark 27 *It was shown in Theorem 26 that $R(M)$ domainlike $\Rightarrow$ R is domainlike. The converse is shown to be false by the following example.*

Example 28 *Let $R = \mathbb{Z}$ and consider $\mathbb{Z}(\mathbb{Z}_2)$. Clearly $\mathbb{Z}$ is domainlike, but $\mathbb{Z}(\mathbb{Z}_2)$ is not domainlike since $(0,1)(2,1) = (0,0)$ and $(0,1) \neq (0,0)$ yet $(2,1)$ is not nilpotent - hence the zero ideal is not primary in $\mathbb{Z}(\mathbb{Z}_2)$.*

Lemma 29 *Let R be Noetherian. Then R is domainlike if and only if $R[[x]]$ is domainlike.*

Proof. ($\Leftarrow$) Suppose $ab = 0$ in R. Then, $ab = 0$ in $R[[x]]$, and $a^n = 0$ for some n.

($\Rightarrow$) Let $f = \sum a_i x^i \in Z(R[[x]])$. Since R is Noetherian, there exists an $r \in R$ such that $rf = 0$. Hence, $ra_i = 0$ for all i. Since R is domainlike and $a_i \in Z(R)$, we have $a_i \in \mathrm{nil}(R)$ for all i, and again since R is Noetherian, $f \in \mathrm{nil}(R[[x]])$. ■

Example 30 *Although $R[[y]]$ domainlike implies that R is domainlike, the converse is not true in general. Consider the following:*

$$R = \mathbb{Z}[x_1, x_2, ...] / \left(x_1, x_2^2, x_3^3, ..., x_n^n, ..., x_2x_3, x_2x_4, ..., x_2x_k, ...\right)$$

Then, R is domainlike, but $R[[y]]$ is not domainlike. To see this, let $f \in R$, if f has no constant term it will be nilpotent. If f has a constant term, it cannot be a zero-divisor. Hence, $Z(R) \subseteq \mathrm{nil}(R)$, and R is domainlike. However, let $g = \overline{x}_3 + \overline{x}_4 y + \overline{x}_5 y^2 + \overline{x}_6 y^3 + \cdots \in R[[y]]$. Then, $\overline{x}_2 g = 0$, but g is not nilpotent. Hence, $Z(R[[y]]) \nsubseteq \mathrm{nil}(R[[y]])$, and $R[[y]]$ is not domainlike.

Theorem 31 *Any localization of a domainlike ring is domainlike.*

Proof. Let R be domainlike and S be a multiplicatively closed set in R. If S contains a zero divisor r, then r is nilpotent (since $Z(R) \subseteq \mathrm{nil}(R)$) and S contains 0. Thus, $R_S = \{0\}$, which is trivially domainlike. So, suppose S contains no zero-divisors and assume $\frac{a}{s_1} \cdot \frac{b}{s_2} = \frac{0}{1}$ with $\frac{a}{s_1} \neq \frac{0}{1}$. Then, there exists a $t \in S$ such that $t(ab - s_1 s_2 \cdot 0) = 0$. Since t is a regular element of R, we get $ab = 0$. Since $\frac{a}{s_1} \neq \frac{0}{1}$, $a \neq 0$, and $b^n = 0$ since (0) is primary. Hence, $\left(\frac{b}{s_2}\right)^n = \frac{0}{1}$, and $(0)_S$ is primary. Thus, R_S is domainlike. ■

Remark 32 *From Theorem 1-3 of Bouvier's paper "Présimplifiable Rings" of 1974 we get the following connections: $R[x_1, x_2, ..., x_n]$ is présimplifiable $\Leftrightarrow$ $R[x_1, x_2, ..., x_n]$ domainlike $\Leftrightarrow$ R is domainlike since Bouvier's definition of domainlike is primary. Bouvier also mentions in Theorem 3 that if R is Artinian, then R is présimplifiable $\Leftrightarrow$ R is local. However a zero-dimensional ring need not be présimplifiable.*

Example 33 *R zero-dimensional and R not présimplifiable. Let $R = \mathbb{Z}_3[x] / (x^2 - 1)$. From Hutchin's example book (example 137), R is 0-dimensional, Noetherian, and non-local. However, $U(R) = \{1, -1, x, -x\}$ so consider $(x+1)(-x-1) = x+1$ and $x+1$ is not a unit. So R is not présimplifiable. It is straightforward to show that R is associate.*

Remark 34 *1. R présimplifiable* $\not\Rightarrow$ *R is superassociate. Any ring can be written as a subring of a direct product of présimplifiable rings - such as* R_M *where M is a maximal ideal. If R présimplifiable* $\Rightarrow$ *R is superassociate, then* R_M *would be superassociate and hence* $\prod R_M$ *is superassociate which would imply that R is associate for every R.*

2. ***This is false:*** *R quasi-local* $\not\Rightarrow$ *R is superassociate. To see this, let* R' *be a quasi-local ring, let* F_1 *be the quotient field of* $\mathbb{Z}[Y]$*, and let* F_2 *be the quotient field of* $\mathbb{Z}[X]$*. Then* $R = R' \oplus F_1 \oplus F_2 \oplus F_2$ *is a quasi-local ring, with subring* $0 \oplus \mathbb{Z}[Y] \oplus \mathbb{Z}[X] \oplus \mathbb{Z}[X] \cong \mathbb{Z}[Y] \oplus \mathbb{Z}[X] \oplus \mathbb{Z}[X]$*. By [Remark 2, Principal Ideals and Associate Rings],* $\mathbb{Z}[Y] \oplus \mathbb{Z}[X] \oplus \mathbb{Z}[X]$ *is not superassociate, and therefore R is not superassociate.*

3. *R superassociate* $\not\Rightarrow$ *R présimplifiable, R is domainlike, or that R is quasi-local. For example, consider the ring* $\mathbb{Z} \times \mathbb{Z}$*. To see that* $\mathbb{Z} \times \mathbb{Z}$ *is superassociate, let A be a subring of* $\mathbb{Z} \times \mathbb{Z}$*, and suppose* $(a_1, b_1), (a_2, b_2) \in A$ *such that* $(a_1, b_1) \sim (a_2, b_2)$*. Then, there exist* $(c_1, d_1), (c_2, d_2) \in A$ *such that* $(a_1, b_1)(c_1, d_1) = (a_2, b_2)$ *and* $(a_2, b_2)(c_2, d_2) = (a_1, b_1)$*. Then* $a_2c_1c_2 = a_2$ *and* $b_2d_1d_2 = b_2$*. If* $a_1,\ b_1 \neq 0$*, then* $c_1c_2 = 1$ *and* $d_1d_2 = 1$*. Clearly, if* $(c_1, d_1) = (1, 1)$ *or* $(-1, -1)$*, then* $(c_1, d_1) \in U(A)$*. If* $(c_1, d_1) = (1, -1)$*,* $(c_1, d_1)^2 = (1, 1)$ *so* $(c_1, d_1) \in U(A)$*. Therefore A is associate. WLOG, if* $a_1 = 0$*, then choose* $c_1 = c_2 = 1$*. Since* $b_2 \neq 0$*, we have that* d_1 *and* d_2 *are either 1 or -1. Therefore, in either case,* $(c_1, d_1), (c_2, d_2) \in U(A)$*, and A is associate. However, a direct product of présimplifiable rings is never présimplifiable and a direct product of domainlike rings is never domainlike.*

4. *Being superassociate is not preserved by direct products or subdirect products. (Remark 2, Principal Ideals and Associate Rings).*

Theorem 35 *R domainlike* $\Rightarrow R/\sqrt{0}$ *is an integral domain (and therefore an associate ring).*

Proof. R domainlike $\Rightarrow (0)$ is primary. Let $ab \in \sqrt{0}$ and $a \notin \sqrt{0} \Rightarrow a^n \neq 0$ for every n, but $a^m b^m = 0$ and $a^m \neq 0$ so $(b^m)^l = 0 \Rightarrow b \in \sqrt{0} \Rightarrow \sqrt{0}$ is prime. Thus $R/\sqrt{0}$ is an integral domain. ■

Example 36 *The converse is false. As before (Example 11) , take* $R = K[[x,y]]/(x)(x,y)$. *R is not domainlike, but* $\sqrt{0} = (\bar{x})$ *which is prime. (Note that* $\sqrt{0}$ *is prime, but* (0) *is not primary since R is not domainlike).*

Remark 37 $R/\sqrt{0}$ *is domainlike* $\Leftrightarrow$ $R/\sqrt{0}$ *is an integral domain* $\Leftrightarrow$ $\sqrt{0}$ *is prime.*

Theorem 38 $R/\sqrt{0}$ *is présimplifiable* $\Leftrightarrow$ $\left(xy = x \text{ and } x \notin \sqrt{0} \Rightarrow y \in U(R)\right)$.

Proof. ($\Rightarrow$) Suppose $xy = x$ and $x \notin \sqrt{0}$. Therefore $\bar{x}\bar{y} = \bar{x}$ and $\bar{x} \neq \bar{0} \Rightarrow \bar{y} \in U\left(R/\sqrt{0}\right) \Rightarrow y \in U(R)$.

($\Leftarrow$) Suppose $\bar{x}\bar{y} = \bar{x}$ and $\bar{x} \neq \bar{0}$. So $x \notin \sqrt{0}$. Then,

$$\begin{aligned} xy - x \in \sqrt{0} &\Rightarrow [x(1-y)]^n = 0 \\ &\Rightarrow x^n (1-y)^n = 0 \\ &\Rightarrow x^n \left(1 - y \cdot \sum_{i=0}^{n} (-1)^i \tbinom{n}{i+1} y^i\right) = 0 \\ &\Rightarrow x^n = x^n y \cdot \sum_{i=0}^{n} (-1)^i \tbinom{n}{i+1} y^i \end{aligned}$$

Since $x \notin \sqrt{0}$, $x^n \neq 0$, and so

$$y \cdot \sum_{i=0}^{n} (-1)^i \tbinom{n}{i+1} y^i \in U(R)$$

Hence, $y \in U(R)$, and $\bar{y} \in U\left(R/\sqrt{0}\right)$. ■

Thus, R présimplifiable implies that $R/\sqrt{0}$ is présimplifiable and hence associate.

Example 39 $R/\sqrt{0}$ *présimplifiable* $\not\Rightarrow$ *R présimplifiable. To see this, consider* $R = \mathbb{Z}(\mathbb{Z}_8)$. *R is not a présimplifiable ring by Theorem 19 and Proposition 20. Then,* $nil(\mathbb{Z}(\mathbb{Z}_8)) = \{(0,a) \mid a \in \mathbb{Z}_8\}$. *Using Theorem 38, suppose that* $(a,b)(y_1,y_2) = (a,b)$ *and* $(a,b) \notin nil(\mathbb{Z}(\mathbb{Z}_8))$ *- i.e.* $a \neq 0$. *Thus* $(ay_1, ay_1 + by_2) = (a,b) \Rightarrow ay_1 = a \Rightarrow y_1 = 1 \Rightarrow (y_1,y_2) \in U(\mathbb{Z}(\mathbb{Z}_8))$. *Thus* $R/\sqrt{0}$ *is présimplifiable by Theorem 38. So,* $R/\sqrt{0}$ *présimplifiable* *does not imply that R is présimplifiable. Further, $R/\sqrt{0}$ présimplifiable does not imply that R is associate, since $\mathbb{Z}(\mathbb{Z}_8)$ is not associate by Theorem 19 and Proposition 20. In addition, it is interesting to note that R associate $\not\Rightarrow R/\sqrt{0}$ is présimplifiable. For example, $R = \mathbb{Z}_3 \times \mathbb{Z}_3$ is associate and $(1,0)(1,0) = (1,0)$ where $(1,0)^n \neq (0,0)$ and $(1,0) \notin U(\mathbb{Z}_3 \times \mathbb{Z}_3)$.*

Theorem 40 *R présimplifiable and (0) not primary $\Rightarrow$ dim$(R) \geq 1$.*

Proof. Since (0) is not primary, there exists $x, y \in R$ such that $xy = 0$ where $y \neq 0$ and $x^n \neq 0$ for all n. Hence, $x \in Z(R) \subseteq J(R)$, and x is contained in every maximal ideal of R.

Now, $S = \{x^n\}_{n=1}^{\infty}$ is a multiplicatively closed set and $x \notin \sqrt{0}$, we have (0) is an ideal disjoint from S. Expand (0) to a prime ideal P disjoint from S. Then, $x \notin P$, and hence P is not maximal. Thus, we have $P \subsetneq M$ for some maximal ideal M of R, and $\dim(R) \geq 1$. ■

Lemma 41 *For any ideal Q, $\sqrt{Q}$ maximal $\Rightarrow Q$ is primary.*

Proof. Let Q be an ideal of R and suppose $\sqrt{Q} = M$, where M is a maximal ideal of R. Let $ab \in Q$ and suppose $a \notin \sqrt{Q}$. Then, to show Q is primary, we must show that $b \in Q$. Since $a \notin \sqrt{Q} = M$, $(M, a) = R$, so $1 \in (M, a)$. Then $1 = m + ra$ with $m \in M$ and $r \in R$. Since $m \in M$, $m^n \in Q$ for some $n \in \mathbb{N}^*$. Now, $1 = (m + ra)^n = m^n + ta$ where $t \in R$, so $b = 1b = (m^n + ta)b \in Q$ since $m^n, ab \in Q$. ■

Remark 42 *If $R/\sqrt{0}$ is a field, then $\sqrt{0}$ is maximal, so (0) is primary by Lemma 41. Hence, R is domainlike. But, $\mathbb{Z}$ is domainlike and $\mathbb{Z}/\sqrt{0}$ is not a field.*

The next example shows that $R/\sqrt{0}$ associate need not imply that R is associate.

Example 43 *By Example 39, $R/\sqrt{0}$ is présimplifiable where $R = \mathbb{Z}(\mathbb{Z}_8)$ and hence $R/\sqrt{0}$ is associate. However, $\mathbb{Z}(\mathbb{Z}_8)$ is not associate.*

Remark 44 *Recall that a ring R is primary if it has a unique prime ideal. By Proposition 7, p. 35, in "Lectures on Rings and Modules", R is primary if and only if R is domainlike and all nonunits are zero divisors.*

Theorem 45 *R présimplifiable and dim*$(R) = 0 \Leftrightarrow R$ *primary.*

Proof. ($\Rightarrow$) Since $\dim(R) = 0$, all prime ideals are maximal. Since R is présimplifiable and $\dim(R) = 0$, we must have $\sqrt{0}$ prime (by Theorem 40) and hence maximal. Since $\sqrt{0}$ is maximal and is the intersection of all prime ideals of R, we have that $\sqrt{0}$ is the only prime ideal in R. Hence R is primary.

($\Leftarrow$) If R is primary, then $dim(R) = 0$ by definition, and since R is domainlike, R is présimplifiable. ■

Remark 46 *If a ring R is présimplifiable and* $\dim(R) = 0$*, then R is primary and therefore domainlike (Remark 44) .*

Remark 47 *Recall that a special principal ideal ring (SPIR) is a PIR with a unique prime ideal. Thus an SPIR is local and thus présimplifiable, and so associate.*

Theorem 48 *If R is a ring with only finitely many distinct principal ideals, then R is associate.*

Proof. If R is a ring with only finitely many distinct principal ideals, then R can be written as a finite direct product of SPIR's and finite local rings (Axtell's paper, Lemma 3.4 - probably elsewhere also). Then an SPIR is associate and so are local rings, so by Theorem 4, we get that R is associate. ■

Let us recall some of the basic definitions involved in the construction of ultraproducts. Let I be a non-empty and let $P(I) = \{A \mid A \subseteq I\}$. D is a **filter on** I if $D \subseteq P(I)$ and

(a) $\phi \notin D$ and $D \neq \phi$,
(b) $A, B \in D$ implies $A \cap B \in D$, and
(c) $A \in D$ and $A \subseteq B$ implies $B \in D$.

A filter D on I is an **ultrafilter** iff for every $A \subseteq I$ either $A \in D$ or $I \setminus A \in D$ - and not both by (a) and (b). Now, let $\{R_\alpha\}_{\alpha \in I}$ be a collection of commutative rings with 1. Let F be an ultrafilter on I. The **ultraproduct of the R_α's modulo** F, $\prod R_\alpha / F$, is defined as $\prod R_\alpha / \sim$ where $(a_i) \sim (b_i)$ if $\{i \in I \mid a_i = b_i\} \in F$. We recall the Los' Property applied to first-order sentences, which essentially states that an ultraproduct of R_α's modulo F will have a given property, A, iff $\{\alpha \in I \mid R_\alpha \text{ has property } A\} \in F$.

Theorem 49 *(Los' Property) If F is an ultrafilter on I and $U = \prod U_i/F$ an ultraproduct, then for any first-order sentence σ, $U \models \sigma$ iff $\{i \in I \mid U_i \models \sigma\} \in F$.*

Theorem 50 *Let $\{R_\alpha\}_{\alpha \in I}$ be a collection of commutative rings with 1. Let F be an ultrafilter on I. Then $\prod R_\alpha/F$ is associate (présimplifiable) $\Leftrightarrow$ $\{\alpha \in I \mid R_\alpha$ is associate (présimplifiable)$\} \in F$.*

Proof. The property of a ring R being associate can be expressed in terms of the first-order sentence

$\sigma_{assoc} = \forall x \forall y \exists z \exists w \exists u \exists v \exists k \forall l[((xz = y) \wedge (yw = x)) \Rightarrow$

$((kl = l) \wedge (uv = k) \wedge (xu = y))]$. Thus the Los' Property gives the desired result. For présimplifiable, use $\sigma_{pré} = \forall x \forall y \exists w \exists v \forall z[(xy = x) \Rightarrow (((x = w) \wedge (wz = w)) \vee ((yu = v) \wedge (vz = z)))]$. ■

Corollary 51 *An ultraproduct of associate (présimplifiable) rings is associate (présimplifiable).*

Proof. For any ultrafilter F on any set I, $I \in F$. ■

Remark 52 *The properties of being domainlike and superassociate are not expressible as first-order sentences, hence Los' Property may not be applied to these characterizations.*

Theorem 53 *Let I be an indexing set. For each $i \in I$ let R_i be a ring from the set $\{R_1, R_2, ..., R_m\}$. Let F be any ultrafilter over I. If R_i is domainlike for every $i \in I$ then $\prod R_i/F$ is domainlike.*

Proof. Suppose $(a_i)(b_i) = (0)$ in $\prod R_i/F$. So, $\{i \mid a_i b_i = 0\} \in F$. If $(a_i) \neq (0)$ then $\{i \mid a_i = 0\} \notin F \Rightarrow \{i \mid a_i \neq 0\} \in F$ since F is an ultrafilter. Now, $a_i b_i = 0$ and $a_i \neq 0 \Rightarrow \exists n_i \in \mathbb{N}$ such that $b_i^{n_i} = 0$ in R_i since R_i is domainlike. Therefore $\{i \mid a_i \neq 0\} \subset \{i \mid b_i^{n_i} = 0$ for some $n_i\} \in F$. Let $n = \max\{n_i\}_{i=1}^m$ then $(b_i)^n = 0$ since

$\{i \mid b_i^{n_i} = 0$ for some $n_i\} \subset \{i \mid b_i^n = 0\} \in F$. ■

The converse to the above theorem is not necessarily true.

Example 54 *Let $I = \{1,2\}$ with ultrafilter $F = \{\{1\}, \{1,2\}\}$ and let $R_1 = \mathbb{Z}_4$ and $R_2 = \mathbb{Z}_6$. Observe that $\prod R_i/F$ is domainlike since the nonzero zero divisors of $\prod R_i/F$ are of the form (a,b) where $0 \neq a \in Z(R_1)$, yet any such (a,b) is nilpotent since R_1 is domainlike, and yet R_2 is not domainlike.*

It is also interesting to note note an arbitrary ultraproduct of domainlike rings need not be domainlike.

Example 55 *Let $R_i = \mathbb{Z}[x]/(x^i)$ for $i \in I = \{2,3,4,5,...\}$. R_i is domainlike since $Z(R_i) = \{xp(x) + (x^i)\} \subset nil(R_i)$. Let F be any ultrafilter over I containing only sets of infinite cardinality, for instance $F = \{\{n, n+1, n+2, ...\} \mid n \geq 2\}$. Now, $(x,x,x,...)(x,x^2,x^3,...) = (0)$ and $(x,x^2,x^3,...) \neq (0)$. However, $\forall n \in \mathbb{N}$, $(x,x,x,...)^n = (x^n,x^n,x^n,...) \neq (0)$ by the construction of our ultrafilter. Thus, $\prod R_i/F$ is not domainlike.*

Recall that a filter, D, on I is called **principal** if for some $A \subseteq I$, $D = \{B \mid A \subseteq B \subseteq I\}$. We call A the generator of the filter. If $A = \{i\}$ for some $i \in I$, then we call i the **base element** of the principal filter. It is straightforward to verify that if the set I is finite, then any ultrafilter, F, on I is principal and has a base element.

Theorem 56 *Let I be an indexing set. Let F be any principal ultrafilter on I with a base element. For every $i \in I$, let R_i be some commutative ring with unity. $\prod R_i/F$ is superassociate $\Leftrightarrow$ R_j is superassociate where j is the base element of the ultrafilter F.*

Proof. WLOG let $j = 1$ be the base element of our ultrafilter F.

($\Rightarrow$) Suppose that R_1 is not superassociate and has non-associate subring S_1. Consider the subring of $\prod R_i/F$ given by $A = \{(a) \in \prod R_i/F \mid$ the R_1-component is from S Since this is a subring of the direct product $\prod R_i$ it's image is also a subring of the ultraproduct $\prod R_i/F$. Thus A is associate by hypothesis. Let $a \sim b$ in S_1 such that $\forall u \in U(R_1)$, $au \neq b$. Now, $(a,1,1,...) \sim (b,1,1,...)$ in A, and A associate implies that there exists a unit, (u) of A such that $(a,1,1,...)(u) = (b,1,1,...)$. Then $(u) \in U(A) \Rightarrow \exists (v) \in U(A)$ such that $(u)(v) = (1) = (1,1,...)$ in A. Therefore $C = \{i \mid u_i v_i = 1_{R_i}\} \in F$. If $1 \notin C$ then $\{1\} \cap C = \emptyset \in F$. This is obviously a contradiction. Thus, any unit of

A contains a unit of S_1 in its first component. So, $(a, 1, 1, ...) \sim (b, 1, 1, ...)$ in A and A associate implies $\exists u_1 \in U(S_1) \subseteq U(R_1)$ such that $au_1 = b$. Contradiction. Thus R_1 is superassociate.

($\Leftarrow$) Assume R_1 is superassociate. Suppose that $\prod R_i/F$ is not superassociate. Let S be a non-associate subgroup of $\prod R_i/F$ and let $(a), (b) \in S$ which are associate and not separated by a unit of S. Let
$S_1 = \{a_1 \in R_1 \mid a_1 \text{ is the first component of some element of } S\}$. Since 1 is the base element of our ultrafilter, we see that S_1 is a subring of R_1 and is hence associate. Again, since 1 is the base element of our ultrafilter, $(a) \sim (b) \Rightarrow a_1 \sim b_1$ in S_1. S_1 associate implies that there exists some $u_1 \in U(S_1) \subseteq U(R_1)$ such that $a_1 u_1 = b_1$. Observe that in our ultraproduct, $(u_1, 0, 0, ...) \in U(S)$ and $(a)(u_1, 0, 0, ...) = (b)$. Contradiction. ■

Corollary 57 *Let F be a principal ultrafilter whose generator, A, is a set of finte cardinality. $\prod R_i/F$ is superassociate $\Leftrightarrow$ R_j is superassociate for every $j \in A$.*

Proof. The proof of Theorem 56 can be easily generalized to show this result. ■

As Theorem 56 suggests, even if we consider more general ultrafilters than principal ultrafilters, we see that an ultraproduct being superassociate does not imply that each constituent ring need be superassociate. The following example illustrates this.

Example 58 *Let $I = \mathbb{N}$ and let our ultrafilter, F, on I be the ultrafilter containing the filter $D = \{\{n, n+1, n+2, ...\} \mid n \in \mathbb{N}\}$. Let R_1 be any non-superassociate ring and let R_i be any non-trivial field for $i \neq 1$. It can be observed that $\prod R_i/F$ is superassociate since every element of F is of infinite cardinality.*

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