ZERO-DIVISOR GRAPHS FOR DIRECT PRODUCTS OF COMMUTATIVE RINGS

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ABSTRACT. We recall several results of zero divisor graphs of commutative rings. We then examine the preservation of the diameter of the zero divisor graph of polynomial and power series rings.

1. INTRODUCTION

Istvan Beck first introduced the concept of relating a commutative ring to a graph in [4]. By the definition he gave, every element of the ring R was a vertex in the graph, and two vertices x, y were connected if and only if $xy = 0$. Beck was primarily interested in colourings of the graph; he conjectured that the chromatic number of a ring - that is, the minimal number of colours necessary to colour the ring's graph such that no two adjacent elements have the same colour - is equal to the size of the largest *complete* subgraph of the graph - that is, the largest subgraph G such that for all vertices a, b in G, a is adjacent to b. He also categorized all finite rings with chromatic number less than four.

In [2], D.D. Anderson and M. Naseer continued working with Beck's definition. They provided a counterexample to his conjecture, but proved several results regarding the cases where the conjecture does hold. They also extended his categorization of finite rings to those with chromatic number less than or equal to four.

A different method of associating a commutative ring to a graph was proposed David F. Anderson and Philip S. Livingston in [1]. They believed that this better illustrated the zero-divisor structure of the ring, and it is the definition used in this paper; to wit:

Definition 1.1. *Zero-Divisor Graph of a Commutative Ring. Let R be a commutative ring. $Z(R)$ is the set of zero divisors of R, and $Z^*(R) = Z(R)\backslash\{0\}$. The zero-divisor graph of R, $\Gamma(Z^*(R))$, usually written $\Gamma(R)$, is a graph in which each element of $Z^*(R)$ is a vertex, and two vertices x and y are connected by an edge if and only if $xy = 0$.*

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Anderson and Livingston's article contains several results that are important for this paper. A graph is *connected* if a path exists between any two vertices in the graph. The *distance* between two vertices a and b is the length of the shortest path between them, declaring the length of each edge to be 1, and is denoted $d(a,b)$; the *diameter* of a graph G is $\sup\{d(a,b) : a$ and b distinct vertices of $G\}$. Anderson and Livingston proved that if R is a commutative ring, then $\Gamma(R)$ is connected and has diameter less than or equal to three.

In [5], S.B. Mulay takes up Anderson and Livingston's definition of the zero-divisor graph and uses it to investigate the cycle structure of $\Gamma(R)$. In [3], M. Axtell, J. Coykendall, and J. Stickles examine the preservation of graph-theoretic properties of the zero-divisor graph under extension to polynomial and power series rings. This paper establishes a set of theorems that describe the diameter of a zero-divisor graph for a direct product $R_1 \times R_2$ with respect to the diameters of the zero-divisor graphs of R_1 and R_2, and also derives some properties of the rings whose zero-divisors are realized as diameter-two graphs.

2. Direct Products

This section provides six theorems regarding the diameters of direct products of commutative rings. The first of these results relies heavily on a theorem from [1] and a corollary thereof.

Theorem 2.1. *(Anderson and Livingston Theorem 2.8) Let R be a commutative ring. Then $\Gamma(R)$ is complete if and only if either $R \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$.*

Corollary 2.2. *Let R be a commutative ring such that diam$(\Gamma(R)) = 1$. Then $R^2 \neq 0$ implies $R \neq Z(R)$.*

Proof. $R^2 \neq 0$ implies that $R \neq Z(R)$ or that there exist $x, y \in Z(R)$ such that $xy \neq 0$. If the first condition is true, then the claim is proven; if the second is true, then by Theorem 2.1 $R \cong \mathbf{Z}_2 \times \mathbf{Z}_2$, and hence $R \neq Z(R)$. □

In the following, for R a commutative ring, let $R^2 = \{ab : a, b \in R\}$. Also, though it be an abuse of notation, let $0 = (0,0)$ as necessary.

Theorem 2.3. *Diameter One by Diameter One*

Let R_1 and R_2 be commutative rings such that diam$(\Gamma(R_1))$ = diam$(\Gamma(R_2)) = 1$. Then:

i) diam$(\Gamma(R_1 \times R_2) = 1$ if and only if $(R_1)^2 = (R_2)^2 = 0$.

ii) diam$(\Gamma(R_1 \times R_2) = 2$ if and only if without loss of generality $(R_1)^2 = 0$ and $(R_2)^2 \neq 0$.

iii) diam$(\Gamma(R_1 \times R_2) = 3$ if and only if $(R_1)^2 \neq 0$ and $(R_2)^2 \neq 0$.

Proof. *i*) ($\Leftarrow$) Let $(R_1)^2 = (R_2)^2 = 0$. Let $x \in R_1$, $y \in R_2$. By Theorem 2.1, it must be that $x_1x_2 = 0$ for all x_1,$x_2 \in R_1$. Likewise for all y_1,$y_2 \in R_2$. Therefore, $(x_1, y_1)\cdot(x_2, y_2) = (x_1x_2, y_1y_2) = 0$ for all $(x_1, y_1), (x_2, y_2) \in R_1 \times R_2$, so $diam(\Gamma(R_1 \times R_2)) = 1$.

($\Rightarrow$) Let $(R_1)^2 \neq 0$. Then, for some $x_1, x_2 \in R_1$, $x_1x_2 \neq 0$. However, this means that $(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2, y_1y_2) \neq 0$ for some $(x_1, y_1), (x_2, y_2) \in R_1 \times R_2$, implying that $diam(\Gamma(R_1 \times R_2)) > 1$.

ii) ($\Leftarrow$) By (*i*), the fact that $(R_2)^2 \neq 0$ implies that $diam(\Gamma(R_1 \times R_2)) > 1$. However, since $(R_1)^2 = 0$, there must exist $x_1 \in R_1$ such that $x_1x_2 = 0$ for all $x_2 \in R_1$. Then $(x_1, 0)$ annihilates any element of $Z^*(R_1 \times R_2)$, and hence a path of length two can be found between any two vertices of $\Gamma(R_1 \times R_2)$ by way of $(x_1, 0)$. So, $diam(\Gamma(R_1 \times R_2)) = 2$.

($\Rightarrow$) Let $(R_1)^2 = (R_2)^2 = 0$. Then by (*i*), $diam(\Gamma(R_1 \times R_2)) = 1$. So, let $(R_1)^2 \neq 0$ and $(R_2)^2 \neq 0$, but assume $diam(\Gamma(R_1 \times R_2)) = 2$. By Corollary 2.2, since $(R_1)^2 \neq 0$, there must exist $m \in R_1 \backslash Z(R_1)$; likewise, there must exist $n \in R_2 \backslash Z(R_2)$. Let $x \in Z(R_1)$, $y \in Z(R_2)$. Consider the elements (m, y) and (x, n) of $Z^*(R_1 \times R_2)$. Since $(m, y) \cdot (x, n) = (mx, ny) \neq 0$, the distance between the vertices is greater than one. Since $diam(\Gamma(R_1 \times R_2)) = 2$, there must be some $(a, b) \in Z^*(R_1 \times R_2)$ such that $(a, b) \cdot (m, y) = (a, b) \cdot (x, n) = 0$. Then $am = ax = 0$, so it must be that $a = 0$, and $by = bn = 0$, so $b = 0$. Then $(a, b) = (0, 0)$, which is not an element of $Z^*(R_1 \times R_2)$. But this is a contradiction. Therefore, it must be that either $(R_1)^2 = 0$ and $(R_2)^2 \neq 0$ or $(R_1)^2 \neq 0$ and $(R_2)^2 = 0$.

iii) By (*i*) and (*ii*). □

Some of the following results make use of Lemma 3.1, proven in the next section, which shows that if $diam(\Gamma(R)) = 2$ and $R = Z(R)$, then for all $e, f \in R$ there exists a $g \in Z^*(R)$ such that $eg = fg = 0$.

Theorem 2.4. *Diameter One by Diameter Two*

Let R_1, R_2 be commutative rings such that $diam(\Gamma(R_1)) = 1$, $diam(\Gamma(R_2)) = 2$. Then:

i) $diam(\Gamma(R_1 \times R_2)) \neq 1$.

ii) $diam(\Gamma(R_1 \times R_2)) = 2$ if and only if $R_1 = Z(R_1)$ or $R_2 = Z(R_2)$.

iii) $diam(\Gamma(R_1 \times R_2)) = 3$ if and only if $R_1 \neq Z(R_1)$ and $R_2 \neq Z(R_2)$.

Proof. *i*) Since $diam(\Gamma(R_2)) = 2$, there must exist $y_1, y_2 \in Z^*(R_2)$, $y_1 \neq y_2$, with $y_1y_2 \neq 0$. Let $x \in Z^*(R_1)$. $(x, y_1) \cdot (x, y_2) = (x^2, y_1y_2) \neq 0$, therefore $diam(\Gamma(R_1 \times R_2)) > 1$.

ii) ($\Leftarrow$) Let $R_1 = Z(R_1)$. Let $x_1 \in Z^*(R_1)$. Then, by Corollary 2.2, $(x_1, 0)$ annihilates any element of $Z^*(R_1 \times R_2)$, and hence $diam(\Gamma(R_1 \times R_2)) \leq 2$. By (*i*), $diam(\Gamma(R_1 \times R_2)) > 1$, so it must be that $diam(\Gamma(R_1 \times R_2)) = 2$. Now, let $R_2 = Z(R_2)$. Assume (x_1, y_1) and (x_2, y_2) are distinct elements of $Z^*(R_1 \times R_2)$ and assume $(x_1, y_1)(x_2, y_2) \neq (0, 0)$. By Lemma 3.1 there exists $y_3 \in Z^*(R_2)$

such that $y_1y_3 = y_2y_3 = 0$. Observe that $(0, y_3) \neq (x_1, y_1)$ and $(0, y_3) \neq (x_2, y_2)$ else $(x_1, y_1)(x_2, y_2) = (0, 0)$. Thus we have $(x_1, y_1) - (0, y_3) - (x_2, y_2)$ and so $diam(\Gamma(R_1 \times R_2)) \leq 2$. By i), $diam(\Gamma(R_1 \times R_2)) = 2$.

($\Rightarrow$) Assume that $R_1 \neq Z(R_1)$ and $R_2 \neq Z(R_2)$ but $diam(\Gamma(R_1 \times R_2)) = 2$. Let $x \in Z(R_1)$, $y \in Z(R_2)$, $m \in R_1 \backslash Z(R_1)$, $n \in R_2 \backslash Z(R_2)$. Since $diam(\Gamma(R_1 \times R_2)) = 2$, there must exist $(a, b) \in Z^*(R_1 \times R_2)$ such that $(x, n)(a, b) = (m, y)(a, b) = 0$. Then $xa = ma = 0$, so $a = 0$, and $nb = yb = 0$, so $b = 0$. Therefore $(a, b) = (0, 0)$, but since $(0, 0) \notin Z^*(R_1 \times R_2)$, this is a contradiction.

iii) By (*i*) and (*ii*). □

Theorem 2.5. *Diameter One by Diameter Three*

Let R_1, R_2 be commutative rings such that $diam(\Gamma(R_1)) = 1$, $diam(\Gamma(R_2)) = 3$. Then:

i) $diam(\Gamma(R_1 \times R_2)) \neq 1$.

ii) $diam(\Gamma(R_1 \times R_2)) = 2$ if and only if $R_1 = Z(R_1)$.

iii) $diam(\Gamma(R_1 \times R_2)) = 3$ if and only if $R_1 \neq Z(R_1)$.

Proof. *i*) Since $diam(\Gamma(R_2)) = 3$, there must exist $y_1, y_2 \in Z^*(R_2)$, $y_1 \neq y_2$, such that $y_1y_2 \neq 0$. Let $x \in Z^*(R_1)$. $(x, y_1) \cdot (x, y_2) = (x^2, y_1y_2) \neq 0$, therefore $diam(\Gamma(R_1 \times R_2)) > 1$.

ii) ($\Leftarrow$) Let $R_1 = Z(R_1)$. Thus $R_1 \neq \mathbf{Z}_2 \times \mathbf{Z}_2$ and so $x_1x_2 = 0$ for all $x_1, x_2 \in R_1 = Z(R_1)$ by Theorem 2.8 of **Anderson/Livingston.** Let $(x_1, y_1), (x_2, y_2) \in Z^*(R_1 \times R_2)$ and assume $(x_1, y_1)(x_2, y_2) \neq (0, 0)$. Thus $y_1y_2 \neq 0$ and so $y_1 \neq 0$ and $y_2 \neq 0$. It is then clear that we have $(x_1, y_1) - (x_1, 0) - (x_2, y_2)$. Using i), $diam(\Gamma(R_1 \times R_2)) = 2$.

($\Rightarrow$) Assume that $R_1 \neq Z(R_1)$ but $diam(\Gamma(R_1 \times R_2)) = 2$. Let $x \in Z^*(R_1)$, $y \in Z^*(R_2)$, $m \in R_1 \backslash Z^*(R_1)$. Since $diam(\Gamma(R_2)) = 3$, there must exist distinct y_1, $y_2 \in Z^*(R_2)$, $y_1 \neq y_2$, $y_1y_2 \neq 0$, such that there does not exist $y_3 \in Z^*(R_2)$, with $y_1y_3 = y_2y_3 = 0$. Consider (m, y_1) and (m, y_2). Assume that there exists (a, b) such that $(m, y_1) \cdot (a, b) = (m, y_2) \cdot (a, b) = 0$. Then $a = 0$ because $ma = 0$, therefore we must have $b \in Z^*(R_2)$ such that $y_1b = y_2b = 0$. However, we have already posited that no such b may exist, so this is a contradiction.

iii) By (*i*) and (*ii*). □

Theorem 2.6. *Diameter Two by Diameter Two*

Let R_1, R_2 be commutative rings such that $diam(\Gamma(R_1)) = diam(\Gamma(R_2)) = 2$. Then:

i) $diam(\Gamma(R_1 \times R_2)) \neq 1$.

ii) $diam(\Gamma(R_1 \times R_2)) = 2$ if and only if $R_1 = Z(R_1)$ or $R_2 = Z(R_2)$.

iii) $diam(\Gamma(R_1 \times R_2)) = 3$ if and only if $R_1 \neq Z(R_1)$ and $R_2 \neq Z(R_2)$.

Proof. *i*) Since $diam(\Gamma(R_1)) = 2$, there must exist $x_1, x_2 \in Z^*(R_1)$, $x_1 \neq x_2$, such that $x_1x_2 \neq 0$. Let $y \in Z^*(R_2)$. $(x_1, y)\cdot(x_2, y) = (x_1x_2, y^2) \neq 0$, therefore $diam(\Gamma(R_1 \times R_2)) > 1$.

ii) ($\Leftarrow$) Without loss of generality let $R_1 = Z(R_1)$. Let $x \in Z^*(R_1)$, $y \in Z^*(R_2)$. Since $R_1 = Z(R_1)$, by Lemma 3.1, for all $x_1, x_2 \in Z(R)$, there exists an x_3 such that $x_3x_1 = x_3x_2 = 0$. So, for any $(x_1, y_1), (x_2, y_2) \in Z^*(R_1 \times R_2)$, there exists $(x_3, 0) \in Z^*(R_1 \times R_2)$ such that $(x_1, y_1)(x_3, 0) = (x_2, y_2)(x_3, 0) = 0$. If, without loss of generality, $(x_2, y_2) = (x_3, 0)$ then we would have $(x_1, y_1)(x_2, y_2) = 0$. Thus, $diam(\Gamma(R_1 \times R_2)) \leq 2$. By (*i*), it must be that $diam(\Gamma(R_1 \times R_2)) = 2$.

($\Rightarrow$) Assume $R_1 \neq Z(R_1)$, $R_2 \neq Z(R_2)$, but $diam(\Gamma(R_1 \times R_2)) = 2$. Let $x \in Z^*(R_1)$, $y \in Z^*(R_2)$, $m \in R_1\backslash Z(R_1)$, $n \in R_2\backslash Z(R_2)$. Consider (x, n) and (m, y). Since $(x, n) \cdot (m, y) = (mx, ny) \neq (0, 0)$ and $diam(\Gamma(R_1 \times R_2)) = 2$, there must be some $(a, b) \in Z^*(R_1 \times R_2)$ such that $(a, b) \cdot (x, n) = (a, b) \cdot (m, y) = (0, 0)$. Then $ma = nb = 0$, so it must be that $(a, b) = (0, 0)$, but $(0, 0) \notin Z^*(R_1 \times R_2)$, a contradiction.

iii) By (*i*) and (*ii*). □

Theorem 2.7. *Diameter Two by Diameter Three*

Let R_1, R_2 be commutative rings such that $diam(\Gamma(R_1)) = 2$ and $diam(\Gamma(R_2)) = 3$. Then:

i) $diam(\Gamma(R_1 \times R_2)) \neq 1$.

ii) $diam(\Gamma(R_1 \times R_2)) = 2$ *if and only if* $R_1 = Z(R_1)$.

iii) $diam(\Gamma(R_1 \times R_2)) = 3$ *if and only if* $R_1 \neq Z(R_1)$.

Proof. *i*) Same as in proof of Theorem 2.6.

ii) ($\Leftarrow$) Same as in proof of Theorem 2.6.

($\Rightarrow$) Assume $R_1 \neq Z(R_1)$ but $diam(\Gamma(R_1 \times R_2)) = 2$. Let $y \in Z(R_2)$, $m \in R_1\backslash Z(R_1)$. Since $diam(\Gamma(R_2)) = 3$, there must exist $y_1, y_2 \in Z^*(R_2)$, $y_1 \neq y_2$, $y_1y_2 \neq 0$ such that there is no $y_3 \in Z^*(R_2)$ with $y_1y_3 = y_2y_3 = 0$. Consider (m, y_1) and (m, y_2). Since $(m, y_1) \cdot (m, y_2) = (m^2, y_1y_2) \neq (0, 0)$ and $diam(\Gamma(R_1 \times R_2)) = 2$, there must exist some $(a, b) \in Z^*(R_1 \times R_2)$ such that $(a, b) \cdot (m, y_1) = (a, b) \cdot (m, y_2) = (0, 0)$. Then $ma = 0$, so $a = 0$. Since $(a, b) \in Z^*(R_1 \times R_2)$, it must be that $b \in Z^*(R_2)$, but $by_1 = by_2 = 0$, and it has already been posited that no such b exists. This is a contradiction.

iii) By (*i*) and (*ii*). □

Theorem 2.8. *Diameter Three by Diameter Three*

Let R_1, R_2 be commutative rings such that $diam(\Gamma(R_1)) = diam(\Gamma(R_2)) = 3$. Then $diam(\Gamma(R_1 \times R_2)) = 3$.

Proof. Since $diam(\Gamma(R_1)) = 3$, there must exist $x_1, x_2 \in Z^*(R_1)$, $x_1 \neq x_2$, $x_1x_2 \neq 0$, such that there is no $x_3 \in Z^*(R_1)$ with $x_1x_3 = x_2x_3 = 0$. Likewise, there must exist $y_1, y_2 \in Z^*(R_2)$ with $y_1y_2 \neq 0$ such that there is no $y_3 \in Z^*(R_2)$ with $y_1y_3 = y_2y_3 = 0$. Consider (x_1, y_1) and (x_2, y_2). Since $(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2, y_1y_2) \neq 0$, $diam(\Gamma(R_1 \times R_2)) > 1$. Assume that $diam(\Gamma(R_1 \times R_2)) = 2$; then there must exist some $(a, b) \in Z^*(R_1 \times R_2)$ such that $(a, b) \cdot (x_1, y_1) = (a, b) \cdot (x_2, y_2) = 0$. Then $ax_1 = ax_2 = 0$, so $a = 0$. Since $(a, b) \in Z^*(R_1 \times R_2)$, it must be that $b \in Z^*(R_2)$, but $by_1 = by_2 = 0$, and it has already been posited that no such b exists. This is a contradiction, so it must be that $diam(\Gamma(R_1 \times R_2)) = 3$. □

If we restrict the rings R_1 and R_2 to be commutative rings with identity, then the previous six theorems yield the following result.

Corollary 2.9. *Let R_1 and R_2 be commutative rings with identity. Then $diam(\Gamma(R_1 \times R_2)) = 3$.*

In a similar manner we consider a commutative ring R with identity has a nontrivial idempotent, e. Due to the idempotent, we can decompose the ring as $R = eR \times (1 - e)R$ and hence $diam(\Gamma(R)) = 3$.

3. Diameter Two Rings

As part of the investigation of direct products, it was helpful to derive some results regarding diameter-two rings similar to those provided by Anderson and Livingston regarding diameter-one rings. The following lemma was cited in the previous section.

Lemma 3.1. *Let R be a commutative ring such that $diam(\Gamma(R)) = 2$ and $Z(R)$ is a subring (not necessarily proper) of R. Then for all $x, y \in Z(R)$, there exists a nonzero z such that $xz = yz = 0$.*

Proof. Let $x, y \in Z(R)$. If $x = 0$, $y = 0$, or $x = y$ then we are done since $diam(\Gamma(R)) = 2$ ensures the existence of the desired element z. So, assume x and y are distinct and nonzero. If $xy \neq 0$ then there exists $z \in Z^*(R)$ such that $xz = yz = 0$ since $diam(\Gamma(R)) = 2$. Thus assume $xy = 0$. Consider $x + y$. Clearly $x + y \neq x$ and $x + y \neq y$. If $x + y = 0$ then $x = -y$ and hence $x^2 = 0$. Thus $z = x$ suffices. So, we assume $x + y \neq 0$. Since $Z(R)$ a subring we have $x + y \in Z^*(R)$. We also assume that $x^2 \neq 0$ and $y^2 \neq 0$ else choose $z = x$ or $z = y$ respectively. Let $X' = \{x' \in Z^*(R) : xx' = 0\}$ and $Y' = \{y' \in Z^*(R) : yy' = 0\}$. Observe that $x \in Y'$ and $y \in X'$, hence X' and Y' are nonempty. If $X' \cap Y' \neq \emptyset$ then choose $z \in X' \cap Y'$. Assume $X' \cap Y' = \emptyset$ and consider $x + y$. Since $x^2 \neq 0$ we see that $x + y \notin X'$ and similarly $x + y \notin Y'$. Since $diam(\Gamma(R)) = 2$ there exists $w \in X'$ such that $x - w - (x + y)$. Then $0 = w(x + y) = wx + wy = wy$ and so $w \in Y'$, a contradiction. □

This excursus into diameter-two graphs leads to results that are interesting in and of themselves, most of which currently relate to $\Gamma(R)$ being complete bipartite, or nearly so. A graph G is complete bipartite if it can be partitioned into two disjoint vertex sets P and Q such that two vertices p and q are connected by an edge if and only if they are in distinct vertex sets. The following lemma makes use of the concept of a graph G having a complete bipartite subgraph G' induced by removing edges from G; to state the idea with more rigor, such a subgraph exists if there exist two disjoint vertex sets P and Q such that, if two vertices p and q that are in distinct vertex sets, then they are connected.

Lemma 3.2. *Let $\Gamma(R)$ be the zero-divisor graph of some commutative ring R. If $\Gamma(R)$ is not complete bipartite, but has a complete bipartite subgraph $\Gamma'(R)$, then $Z(R)$ is a subring of R.*

Proof. Let $a, b \in Z^*(R)$. Clearly $ab \in Z(R)$ and $-a \in Z(R)$.Consider $a+b$. Since it is possible to form a complete bipartite graph by removing edges from $\Gamma(R)$, there must exist nonempty sets P and Q such that $P \cup Q = Z(R)$, $P \cap Q = \emptyset$, and $pq = 0$ for all $p \in P$, $q \in Q$. If $a \in P$ and $b \in P$, then let $q \in Q$. Thus, $q(a+b) = qa+qb = 0+0 = 0$, so $a+b \in Z(R)$. Likewise for $a \in Q$ and $b \in Q$. So, without loss of generality, assume $a \in P$ and $b \in Q$. Since $\Gamma(R)$ is not complete bipartite, there must be an edge that does not connect a vertex of P to a vertex of Q; let it lie between $p_1, p_2 \in P$. $p_1(b+p_2) = p_1 b + p_1 p_2 = 0$, so either $b+p_2 = 0$, $b+p_2 \in Q$, or $b+p_2 \in P$. If $b+p_2 = 0$, then $0 = b0 = b(b+p_2) = b^2 + bp_2 = b^2$, so $b(a+b) = ba + b^2 = 0+0 = 0$ and hence $a+b \in Z(R)$. If $b+p_2 \in Q$, then $0 = a(b+p_2) = ab + ap_2 = ap_2$, so $p_2(a+b) = p_2 a + p_2 b = 0+0 = 0$ and hence $a+b \in Z(R)$. If $b+p_2 \in P$ then for any $q \in Q$ we have $q(b+p_2) = 0$ and $0 = q(b+p_2) = qb + qp_2 = qb$. Thus, $q(a+b) = qa+qb = 0+0 = 0$. □

Using the lemmas above, it is possible to prove a theorem that is nearly an analogue of **Anderson and Livingston** Theorem 2.8 for diameter-two zero-divisor graphs.

Theorem 3.3. *Let R be a commutative ring. If $\Gamma(R)$ is not complete bipartite but a complete bipartite subgraph can be formed by removing edges from $\Gamma(R)$, then for all $x, y \in Z(R)$ there exists a $z \in Z(R)$ such that $xz = yz = 0$.*

Proof. By Lemma 3.2, $Z(R)$ is a subring of R. By Lemma 3.1, since $Z(R)$ is a ring with $Z(R) = Z(Z(R))$ and $diam(\Gamma(Z(R))) = diam(\Gamma(R)) = 2$, then for all $x, y \in Z(R)$ there exists a $z \in Z(R)$ such that $xz = yz = 0$. □

This theorem could be used to give a general description of all diameter-two zero-divisor graphs if the following conjecture could be proven.

Conjecture 3.4. *Let G be a connected graph with diameter two. If it is not possible to form a complete bipartite graph G' by removing edges from G, then $G \neq \Gamma(R)$ for any commutative ring R.*

4. Realizing G as $\Gamma(R)$

A natural question regarding zero-divisor graphs is to ask which graphs can be realized as $\Gamma(R)$ for some commutative ring R. Given a particular graph, it is sometimes possible to prove that it cannot be realized as $\Gamma(R)$, as in the following example, which incidentally lends some creedence to Conjecture 3.4.

Example 4.1.

Let G be a pentagon, and label its vertices consecutively a, b, c, d, e. Mulay and (DeMeyer/Schneider) proved that if a zero-divisor graph has a cylce, it must have girth of 3 or 4. Thus G cannot be realized as $\Gamma(R)$ for some commutative ring R. If we modify this example to include a path from e to b, then we can show that the result is still never a zero-divisor graph of a ring. Consider $a+c$. Since $b(a+c) = 0$ and $d(a+c) \neq 0$, we have that $a+c \in Z^*(R)$. However, it can easily be shown that $a+c$ cannot equal any of the five non-zero zero divisors. For example, if $a+c = b$, the $0 = eb = e(a+c) = ea + ec = 0 + ec$, but $ec \neq 0$.

By introducing the idea of a *looped vertex*, it is possible to give a few more results regarding graphs that cannot be realized as $\Gamma(R)$. A vertex is looped if it corresponds to an element $z \in R$ such that $z^2 = 0$, and is so called because when the graph is drawn, there is an edge that forms a loop in order to connect the looped vertex to itself.

Lemma 4.2. *Let G be a connected graph with two adjacent looped points a and b (that is, $a^2 = b^2 = 0$). If $diam(G) > 1$ and there is no $h \in G$ such that h is adjacent to both a and b, then G cannot be realized as $\Gamma(R)$ for any commutative ring R.*

Proof. Assume that G can be realized as $\Gamma(R)$ for some commutative ring R. Since $diam(G) > 1$, there must be some vertex in G besides a and b connected to either a or b (but not both, by the hypotheses of the theorem); let that vertex be denoted c. Consider the element $a+b$. Trivially, $a+b \neq a$ and $a+b \neq b$. Since $c(a+b) \neq 0$, $a+b \neq 0$. However, $a(a+b) = 0$ implies $a+b \in Z^*(R)$, and thus there is a vertex in G corresponding to $a+b$. But such a point would be adjacent to both a and b, a contradiction. □

A *star graph* is a specific type of complete bipartite graph in which one of the two disjoint vertex sets contains only one element. The lone element is called the *center* of the star graph.

Lemma 4.3. *Let G be a star graph with a non-looped center (that is, a center that does not square to zero). If any vertex in G is looped, then G cannot be realized as $\Gamma(R)$ for any commutative ring R.*

Proof. Assume that G can be realized as $\Gamma(R)$ for some commutative ring R, and let the center of G be denoted b. Assume that some point $a \in G$, $a \neq b$, is looped. Consider the element $a+b$. $a(a+b) = 0$, so $a+b \in Z(R)$, but $b(a+b) = b^2 \neq 0$, so $a+b \neq 0$; therefore, it must be that $a+b \in Z^*(R)$. A contradiction since $bx = 0$ for all $x \in Z(R)\backslash\{b\}$. □

However, the results above only show cases in which G cannot be realized as $\Gamma(R)$; it would be more useful to know in which cases G *can* be realized as $\Gamma(R)$, and in such cases, what can be learned about R upon inspection of G. The following result shows that all diameter-one graphs correspond to a commutative ring, but not necessarily one with identity. It is easily seen that every finite graph G of diameter 1 can be realized as $\Gamma(R)$ for some commutative ring R by letting $R = \mathbf{Z}_{k+1}$ with multiplication defined by $xy = 0$ for all $x, y \in \mathbf{Z}_{k+1}$.

However, if Conjecture 3.4 could be proven, it could be shown that every diameter-two graph G is either complete bipartite (and hence can be realized as $\Gamma(F_a \times F_b)$, F_a and F_b finite fields), or that G can be realized as $\Gamma(R)$ for some commutative ring R if and only if G can be realized as $\Gamma(R')$ for some R' such that $R' = Z(R')$. This makes the problem much simpler, because if $R' = Z(R)$ is finite, one needs only compare it to all of the rings of order $|R'| = |Z(R)|$ (a finite set) in order to determine whether G can be realized as $\Gamma(R)$.

5. Conclusion

The investigation regarding diameters of direct products is essentially finished; however, it acted as a motivation for asking seemingly unrelated questions, most of which remain unanswered, about what the structure of $\Gamma(R)$ tells us about R. This paper approaches an understanding of diameter-two zero-divisor graphs similar to what is already known about diameter-one graphs. However, it does not touch on diameter-three graphs at all. Because of the complexity of diameter-three graphs, studying the images of subrings and ideals in $\Gamma(R)$, and the inverse images of subgraphs of $\Gamma(R)$ in R, might make the problem easier. In [5], Mulay offers a system for decomposing $\Gamma(R)$ into subgraphs that is suggestive. Hopefully, it would eventually be possible to determine whether G can be realized as $\Gamma(R)$ for any G.

Regarding the more general study of what $\Gamma(R)$ reveals about R, it may be helpful to modify the definition of $\Gamma(R)$ so as to contain more information about the nilpotency of the elements of R. The definition used in this paper, which allows rings with distinct zero-divisor structures to be realized as the same zero-divisor graph, is less useful than it could be in, for example, attempting to create a classification of commutative rings based on their zero-divisor graphs. In Section 4 of this paper there is an attempt to include more information by adding 'loops' to some vertices; however, it may be more representative of the structure of the zero-divisors to add more vertices to the graph. Consider, for example, $\mathbf{Z}_2 \times \mathbf{Z}_2$, a diameter-one ring that is a frequent exception to results about diameter-one rings and more often exhibits diameter-two behaviour. If each element of $Z^*(\mathbf{Z}_2 \times \mathbf{Z}_2)$ were represented by two vertices instead of one, then $diam(\Gamma(\mathbf{Z}_2 \times \mathbf{Z}_2)) = 2$; for any diameter-one ring not congruent to $\mathbf{Z}_2 \times \mathbf{Z}_2$, this doubling of vertices would have leave the diameter of $\Gamma(R)$ unchanged. Studying modifications of this sort may also yield theorems that can be used to count things, which are typically very useful.

6. Bibliography

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